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R. P. Boas, Jr., *Executive Editor*

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Mathematical Reviews

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HISTORY

***Gurjar, L. V. Ancient Indian Mathematics and Vedha.** Ideal Book Service, Poona, 1947. vi+202 pp. Rs. 6-4-0 or 10s.

This book consists of two parts, the first dealing with the history of ancient Hindu mathematics and the second dealing with vedha, the methods and instruments of ancient Hindu astronomy. The author divides the Hindu contributions to these sciences into three groups: those of the Vedic age, prior to 750 B.C.; those of the premediaeval age, from 750 B.C. to A.D. 400; and those of the mediaeval age, A.D. 400 to the 17th century. The chapter headings of the two parts are: Part I, introduction, development of logistics in the Vedic age, developments in geometry in the Vedic age, Bakhṣālī manuscript, mathematics of the mediaeval age; Part II, introductory, Vedic age, Sūrya Siddhānta, Āryabhaṭa to Bhāskara and observatories. A list of works consulted is appended.

Many quotations are given from Sanskrit sources to illustrate the state of Hindu mathematical and astronomical knowledge in each of the three ages considered. A check of most of these quotations, except those from the Śulva Sūtras, in text or translation, showed discrepancies, ranging from minor deviations to wide variations, with the reviewer's sources. The minor deviations may be due to typographical errors. A major deviation occurs on pp. 15, 16 in the quotation from the Taittirīya Saṁhitā where powers of 10, including 10^{19}, are listed. A. B. Keith [The Veda of the Black Yajus School Entitled Taittiriya Sanhita, Harvard Oriental Series, 1914, vol. II, p. 585] translates this passage as including 10^{12} as the highest power of 10 and the remainder is not translated as mathematical. Datta and Singh [History of Hindu Mathematics, Lahore, 1935, Part I, p. 9] discuss this same passage and the corresponding one in the White Yajur Veda and state that numbers including 10^{12} are given.

The author's only mathematical representative of the premediaeval period is the Bakhṣālī manuscript. The quotations do not agree with the work of G. R. Kaye [The Bakhshālī Manuscript, Parts I, II, III, Archaeological Survey of India, vol. 43, Calcutta, 1927, 1933]. Also, the author dates this manuscript as not later than A.D. 300, whereas Kaye dates it in the twelfth century.

The author's only astronomical representative of the premediaeval age is the Sūrya Siddhānta. The modern Sūrya Siddhānta, as translated by Burgess [J. Amer. Oriental Soc. 6, 141–498 (1860); reprinted, University of Calcutta, 1935], dates from about A.D. 1100. Varāha Mihira in his Pañcasiddhāntikā quotes an earlier S.S. of about the middle of the 6th century and it is usually thought that the original dates from A.D. 400 at the earliest. The author's quotations are taken from the modern work.

The book contains much interesting information such as detailed discussions on the Hindu method of solving indeterminate equations and sections on astronomical instruments and observatories. However, it does have several defects such as the many typographical errors, the lack of references for many quotations, and the contentious spirit in which it is written.

Some omissions in the works consulted might be noted. No mention is made of D. E. Smith, History of Mathematics, Ginn and Co., 2 vols., 1923, 1925. No notice is taken of The Āryabhaṭīya of Āryabhaṭa by W. E. Clark, Chicago, 1930, although it is a study of the same text and commentary of the Āryabhaṭīya that is cited by the author.

E. B. Allen (Troy, N. Y.).

***Thureau-Dangin, F. Un problème algébrique babylonien.** Halil Edhem Hatıra Kitabı (Recueil Offert à la Mémoire de Halil Edhem), pp. 44–47. Türk Tarih Kurumu Basımevi, Ankara, 1947.

The author presents an analysis of a Babylonian mathematical text (dated around 1700 B.C.) to show that, of the three types of trinomial equations of the second degree distinguished by Al-Khwārizmī (died about 850 A.D.), (1) $x^2+ax=b$, (2) $x^2+b=ax$, (3) $ax+b=x^2$, type (2) is documented in Babylonian mathematics. The presence of (1) and (3) in Babylonian mathematics had been noted previously.

A. J. Sachs (Providence, R. I.).

Thureau-Dangin, F. L'origine de l'algèbre. Acad. Inscriptions et Belles-Lettres. C. R. **1940**, 292–318 (1940).

Rey, Abel. A propos des "Mathématiques Babyloniennes" de M. Thureau-Dangin. Thalès 4, 227–234 (1940).

Neugebauer, O. A table of solstices from Uruk. J. Cuneiform Studies 1, 143–148 (1947).

van der Waerden, B. L. Egyptian "Eternal Tables." I. Nederl. Akad. Wetensch., Proc. **50**, 536–547 (1947).

The reviewer published [Trans. Amer. Philos. Soc. (N.S.) 32, 209–250 (1942); these Rev. 3, 257] Egyptian planetary texts of the Roman period, though without answering the question of their computation. It is shown in the present paper that the Babylonian rules for the computation of planetary ephemerides lead to an explanation of the dates found in the Egyptian texts. This is shown for Venus and Jupiter by reducing all positions given by the text into one single period; the points obtained show the same relation to the longitudes as expected from the Babylonian scheme for the velocities in different parts of the ecliptic. The agreement is very good for Venus, less for Jupiter. For Mars it is shown that the Babylonian method leads to a velocity $vu/(24+u)$ degrees per day if v is the solar velocity and $360+u$ the synodical arc, assumed constant on sections of 60° of the ecliptic. This relation is satisfied by the velocities deducible from the Egyptian tables. *O. Neugebauer.*

***Lundsgaard, Erik. Ægyptisk Matematik. [Egyptian Mathematics].** J. H. Schultz Forlag, Copenhagen, 1945. 39 pp. (Danish) 3.75 Danish crowns.

A lecture, mostly concerning Egyptian arithmetic without going into details. *O. Neugebauer.*

Mineur, Adolphe. De la géométrie grecque. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **31** (1945), 683–710 (1946).

Thaer, Clemens. Euklids Data in arabischer Fassung. Hermes **77**, 197–205 (1942).

***Michaux, Maurice. Le Commentaire de Marinus aux Data d'Euclide. Étude Critique.** Université de Louvain. Recueil de Travaux d'Histoire et de Philologie. Ser. 3, fasc. 25, 1947. 77 pp.

Marinus, the successor of Proclos, is credited with a commentary to Euclid's "Data." The Greek text with Latin translation is published in vol. VI of the Teubner edition of Euclid's works. The present study concerns those who are interested in the Greek theories about existence, constructibility, "actually constructed" and similar concepts. In chapter I, a detailed analysis of the structure of the work and its contents is given. Chapter II contains a French translation. Chapter III puts forward the assumption that this work was not written by Marinus himself but by one of his pupils who attended his lectures. The author also confirms results obtained by Thaer [cf. the preceding title] about the original contents of the Data. *O. Neugebauer.*

Richards, John F. C. Boissière's Pythagorean game, translated with notes on the text. Scripta Math. **12**, 177–217 (1946).

Translation of Boissière's "Ludus Pythagoreus" (Rythmomachia) with notes and list of errata and an abbreviated version of Le Fèvre's description of the game.

F. A. Behrend (Melbourne).

Procissi, Angiolo. Sopra una questione di teoria dei numeri di Guglielmo Libri, ed una lettera inedita di Agostino Cauchy. Boll. Un. Mat. Ital. (3) **2**, 46–51 (1947).

Van den Broek, J. A. Euler's classic paper "On the strength of columns." Amer. J. Phys. **15**, 309–318 (1947).

Bureau, Fl. Le Traité de la Lumière de Christian Huygens. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **32** (1946), 730–744 (1947).

Miller, G. A. Abstract group generated by the quaternion units. Proc. Nat. Acad. Sci. U. S. A. **33**, 235–236 (1947).

The author claims that Hamilton's definition of quaternions in 1843 justifies calling him, rather than Cayley, the founder of the abstract theory of groups.

G. de B. Robinson (Toronto, Ont.).

Karpinski, Louis C. Mathematics in Latin America. A brief survey of their publications to 1850. Scripta Math. **13**, 59–63 (1947).

Dumont, M. Mohammed Ibn Mousa Al-Khowarizmi. Rev. Gén. Sci. Pures Appl. N. S. **54**, no. 2, 7–13 (1947).

Fréchet, Maurice. Biographie du mathématicien alsacien Arbogast. Thalès **4**, 43–55 (1940).

L. F. A. Arbogast lived 1759–1803.

Erdélyi, A. Obituary: Harry Bateman. J. London Math. Soc. **21** (1946), 300–310 (1947).

Etherington, I. M. H. Obituary: George David Birkhoff. Edinburgh Math. Notes no. **36**, 22–23 (1947).

Dickson, L. E. Obituary: Hans Frederik Blichfeldt. 1873–1945. Bull. Amer. Math. Soc. **53**, 882–883 (1947).

***Brahe, Tycho. Tycho Brahe's Description of his Instruments and Scientific Work as given in Astronomiae Instauratae Mechanica (Wandesburgi 1598).** Translated and edited by Hans Raeder, Elis Strömgren and Bengt Strömgren. Det Kongelige Danske Videnskabernes Selskab, Copenhagen, 1946. 144 pp.

Conforto, Fabio. Obituary: Federigo Enriques. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **6**, 226–252 (1947).

Speiser, Andreas. Einteilung der sämtlichen Werke Leonhard Eulers. Comment. Math. Helv. **20**, 288–318 (1947).

This lists Euler's works with their numbers in Eneström's catalogue [cf. Jber. Deutsch. Math. Verein. **19**, *104–116*, *129–142* (1910); 4. Erganzungsband, Teubner, Leipzig, 1910–1913], their original places of publication, and their location in the edition of Euler's collected works.

Beumer, M. G. A historical detail from the life of Gottlob Frege (1848–1925). Postscript by E. W. Beth. Simon Stevin **25**, 146–151 (1947). (Dutch)

Discussion of a letter from Frege concerning Hilbert's foundations of geometry. The letter was published in S.-B. Heidelberger Akad. Wiss. **1940**, no. 6; these Rev. **2**, 306.

Majorana, Quirino. Obituary: Galileo Galilei. Rend. Sess. Accad. Sci. Ist. Bologna (N.S.) **46**, 92–117 (1942).

Conforto, Fabio. Obituary: Giuseppe Gherardelli. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **6**, 215–216 (1947).

Loria, Gino. Ernest de Jonquières, sailor and scientist. Scripta Math. **13**, 5–15 (1 plate) (1947).

Obituary: N. E. Joukovsky (1847–1921). Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **11**, 9–40 (1947).

Pages 27–40 contain a list of Joukovsky's publications.

Yur'ev, B. I. The life and work of N. E. Žukovskiĭ (1847–1921). Bull. Acad. Sci. URSS. Cl. Sci. Tech. [Izvestia Akad. Nauk SSSR] **1947**, 241–252 (1947). (Russian)

Obituary: Nikolaĭ Egorovič Žukovskiĭ (1847–1921). Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **11**, 3–8 (1 plate) (1947). (Russian)

Michieli, Adriano Augusto. Una famiglia di matematici e di poligrafi trivigiani: i Riccati. I. Iacopo Riccati. Ist. Veneto Sci. Lett. Arti. Parte II. Cl. Sci. Mor. Lett. **102**, 535–587 (1944).

Michieli, Adriano Augusto. Una famiglia di matematici e di poligrafi trivigiani: i Riccati. II. Vincenzo Riccati. Ist. Veneto Sci. Lett. Arti. Parte II. Cl. Sci. Mor. Lett. **103**, 69–109 (1944).

Michieli, Adriano Augusto. Una famiglia di matematici e di poligrafi trivigiani: i Riccati. III. Giordano Riccati. Ist. Veneto Sci. Lett. Arti. Parte II. **104**, 771–832 (1946).

Michieli, Adriano Augusto. Una famiglia di matematici e di poligrafi trivigiani: i Riccati. IV. Francesco Riccati. Ist. Veneto Sci. Lett. Arti. Parte II. **104**, 833–859 (1946).

Gagnebin, S. Pour le troisième centenaire de la naissance de Leibniz (1646–1716). Dialectica 1, 77–97 (1947).

★**Hofmann, E. Vom Werden der Leibnizschen Mathematik.** Ber. Math.-Tagung Tübingen 1946, pp. 13–35 (1947).

Myrberg, P. J. Ernst Lindelöf in memoriam. Acta Math. 79, i–iv (1947).

★**The Royal Society. Newton Tercentenary Celebrations, 15–19 July 1946.** Published for The Royal Society at the University Press, Cambridge, 1947. xvi+92 pp. (6 plates).

This volume contains the text of the following lectures: Newton, by Andrade; Newton, the man, by Keynes; Newton and the infinitesimal calculus, by Hadamard; Newton and the atomic theory, by Vavilov; Newton's principles and modern atomic mechanics, by N. Bohr; Newton: the algebraist and geometer, by Turnbull; Newton's contributions to observational astronomy, by W. Adams; Newton and fluid mechanics, by Hunsaker.

Čech, Eduard. The scientific work of Bedřich Pospíšil. Časopis Pěst. Mat. Fys. 72, D1–D9 (1947). (Czech)

Faedo, Sandro. Obituary: Leonida Tonelli. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) 6, 217–225 (1947).

Lebesgue, Henri. L'oeuvre mathématique de Vandermonde. Thalès 4, 28–42 (1940).

Castelnuovo, G. Obituary: Vito Volterra. Mem. Soc. Ital. Sci. (3) 25, 87–95 (1943).

ALGEBRA

Dvoretzky, A., and Motzkin, Th. A problem of arrangements. Duke Math. J. 14, 305–313 (1947).

Let p white and q black balls, $p \geqq \alpha q$, be arranged on a circle. A ball B is termed an α (α^*) head if the number of white balls on any arc beginning with B is more than (at least) α times the number of black balls. The number of α (α^*) heads is denoted by $N(\alpha)$ ($N(\alpha^*)$), its maximum and minimum by $N^+(\alpha)$, $N^-(\alpha)$ ($N^+(\alpha^*)$, $N^-(\alpha^*)$), respectively. The authors show that $N^-(\alpha)=p-[\alpha q]$, $N^+(\alpha)=\min_{r/s<\alpha}(ps-qr)$, $N^-(\alpha^*)=[p-\alpha q+1]$, $N^+(\alpha^*)=\min_{r/s<\alpha}(ps-qr)$. Conditions are derived for $N^-=N^+$. In particular, $N^-=N^+$ if α is an integer. Expressions are derived for N^+ in terms of the expansions of q and p/q into continued fractions. Putting $n=N/(p+q)$ it is further shown that $n(\alpha)>(p-\alpha q)/(p+q)$ for nonintegral α. The asymptotic behavior of n^+ and n^- is investigated. The results are applied to the solution of the following problem. In an election, candidates P and Q receive p and q votes, respectively; required the probability that the ratio of the ballots for P to those for Q will, throughout the counting, be larger than (larger than or equal to) α.

H. B. Mann (Columbus, Ohio).

Kishen, K. On fractional replication of the general symmetrical factorial design. Current Sci. 16, 138–139 (1947).

The fractional replication of general symmetrical factorial designs has been proposed by D. J. Finney [Ann. Eugenics 12, 291–301 (1945); these Rev. 7, 213]. Finney also gave a method of constructing such arrangements for m factors at p levels, where p is a prime. The author outlines a method of obtaining these designs for m factors at p^n levels, where p is a prime. His method is based on the method of R. C. Bose and K. Kishen [Sankhyā 5, 21–36 (1940); these Rev. 4, 222] for obtaining in the same case symmetrical factorial designs in which only interactions of higher order are confounded.

H. B. Mann (Columbus, Ohio).

★**Morand, Max. Introduction Mathématique aux Théories Physiques Modernes. Première Partie. Nombres Complexes, Nombres Hypercomplexes, Matrices, Opérateurs, Applications Élémentaires.** Librairie Vuibert, Paris, 1947. 139 pp. 350 francs.

This first volume is a development of elementary mathematics from the positive integers through some topics in combinatory analysis, vectors, matrices and determinants. Emphasis is placed on those topics relevant to modern physical theories. The first half of the book develops the number system through (unsigned) arithmetic numbers, algebraic (signed) numbers introduced by vector concepts, complex numbers with emphasis on their geometric interpretation [p. 61] and their realization by matrix operators. A definition of "arrangements" begins chapter two and is followed by the representation of permutations by matrices and the geometric interpretation of these matrices. The finding of the irreducible representations of the symmetric group on three symbols is a typical illustration of the point of view of the book. Determinants are defined and solutions of systems of linear equations obtained with their aid. (This third chapter seems to the reviewer the least perspicacious in the book.) The last chapter develops the vector calculus in three dimensions with emphasis on their use in describing transformations of the space. The author promises to include in volume two a discussion of the spinor representation of the orthogonal group. This excellent book might well be studied by able students of physics and mathematics at an early stage in their career.

W. Givens.

Bowker, Albert H. On the norm of a matrix. Ann. Math. Statistics 18, 285–288 (1947).

A function $\varphi(A)$ of the elements of a real matrix A is called a norm if (1) $\varphi(cA)=|c|\varphi(A)$, c a scalar; (2) $\varphi(A+B)\leqq\varphi(A)+\varphi(B)$, if $A+B$ is defined; (3) $\varphi(AB)\leqq\varphi(A)\varphi(B)$, if AB is defined; and $\varphi(e_{ij})=1$, where e_{ij} is a fundamental unit matrix. It is shown that $R(A)=\max_{(i)}\sum_j|a_{ij}|$ is a norm, that no "best possible" norm exists and that $\sum_{i,j}|a_{ij}|$ is the "worst possible" norm. The practical usefulness of these and some other norms is discussed briefly.

W. Givens (Knoxville, Tenn.).

Durand, Émile. Une expression nouvelle des relations entre les 16 grandeurs bilinéaires formées avec les matrices du type Dirac. C. R. Acad. Sci. Paris 225, 280–282 (1947).

The author defines four real 4 by 4 matrices ϵ_p whose products $\epsilon_p\epsilon_q=b_{pq}$ are a basis for the real total matric algebra of order 4^2. Each b_{pq} is the matrix of a bilinear form $\psi^* b_{pq}\psi=[b_{pq}]$. These 16 forms constitute a 4 by 4 matrix of determinant Δ. Let B_{pq} be the minor of $[b_{pq}]$ in Δ.

The author is then able to write certain quadratic identities in the theory of Dirac in the compact form $|\Delta|^{\frac{1}{2}}[b_{pq}]=B_{pq}$.
C. C. MacDuffee (Madison, Wis.).

Durand, Émile. Généralisation des formules d'Olinde Rodrigues et nouvelle représentation des rotations d'Univers. C. R. Acad. Sci. Paris **225**, 375–377 (1947).

Let b_{pq} be defined as in the preceding note, $\psi^*b_{pq}\psi=[b_{pq}]$, where $\psi^*\psi=1$. By specializing the ψ's, the author obtains in turn the formula of Rodrigues for an orthogonal matrix, rotations in space, pure and mixed Lorentzian transformations and, with a change of base, the Cayley-Klein parameters. He determines the relation of his equations to those of Dirac and obtains a new representation of the orthogonal transformations of relativity. *C. C. MacDuffee.*

Papy, Georges. Sur les formes cubiques alternées de rang inférieur à 8. Bull. Soc. Roy. Sci. Liége **15**, 77–83 (1946).

Schouten [Rend. Circ. Mat. Palermo **55**, 137–156 (1931)] gave the complete classification of alternating cubic forms of rank less than 8. This paper gives a simpler demonstration which throws light on the structure of forms of different classes. If in an algebraic form of rank r a certain linear form in the variables is made equal to zero, an algebraic form Ω' of rank r' is obtained. The class of Ω' is called the type of the linear form relative to Ω. The difference in rank, $r-r'$, is called the rank of the linear form. The form Ω is said to admit the linear form of this rank and type. The alternating cubics of a given rank are thus analysed according to the linear forms of maximum class which they admit, upon the types of these forms and upon the number of linearly independent forms of given rank and type.

Using a nomenclature in which the canonical form of an alternating quadratic (linear complex) of rank r is denoted by $11'+22'+33'+\cdots+rr'$ the author states that every alternating cubic of rank 3 is of the form 123: there exist no forms of rank 4, and the forms of rank 5 are equivalent to $1(23+45)$. He then obtains two forms of rank 6:

$$123+456;\quad 123+345+561,$$

and the following five forms of rank 7:

$$1(23+45+67),\qquad 123+345+567,$$
$$1(23+45+67)+(345),\qquad 1(23+67)+246+357,$$
$$1(23+45+67)+246+567.$$

D. E. Littlewood (Cambridge, England).

Littlewood, D. E. On the concomitants of spin tensors. Proc. London Math. Soc. (2) **49**, 307–327 (1947).

In $n=2\nu$ or $2\nu+1$ variables there is an irreducible representation of the orthogonal group corresponding to each partition $(\lambda_1, \lambda_2, \cdots, \lambda_\nu)$ into less than ν parts. There are, however, also two-valued or spin representations corresponding to partitions into ν parts each of which is half of an odd integer. The simplest case is that of the partition $((\frac{1}{2})^\nu)$ which is called the basic spin representation. A tensor which is transformed by the basic spin representation is called a basic spin tensor. A basic spin tensor of type $[(\frac{1}{2})^\nu]$ must have a concomitant of type $[1^\nu]$ which is regarded as reducible, being the principal part of the product of the basic spin tensor by itself. With this convention the author finds the types of irreducible concomitants of a basic spin tensor in case $3\leqq n\leqq 8$ and shows that in none of these cases are there irreducible concomitants of degree greater than 2. He also determines the concomitants of degree 2 for a general value of n and shows that when $n=10$ or 11 irreducible concomitants of degree greater than 2 do exist. He indicates how the concomitants of other types of spin tensors may be obtained and concludes by determining the types of irreducible concomitants of a double wave tensor [A. S. Eddington, Relativity Theory of Protons and Electrons, Cambridge University Press, 1936, p. 154]. The methods used are based on the author's earlier paper [Philos. Trans. Roy. Soc. London. Ser. A. **239**, 305–365 (1944); these Rev. **6**, 41]. *J. Williamson* (Flushing, N. Y.).

Abstract Algebra

***Kiss, Stephen A. Transformations on Lattices and Structures of Logic.** Published by the author, 11 East 92nd Street, New York 28, N. Y., 1947. x+322 pp. $7.50.

This book is intended by the author to be "a guide for those actively interested in the development of 4-class and 16-class logic and mathematics." It consists of nine chapters with the following titles: Structures; Sets, groups and rings; Partially ordered systems and lattices; Link-preserving transformations and direct unions of lattices; Boolean algebras; Direct unions of chains and number systems; Calculus of classes; Calculus of propositions; Calculus of propositional functions.

The author's claim for novelty is based on his result that a "Boolean algebra B^n (of order 2^n) has 2^n mutually distributive set-theoretical systems which can be constructed by means of the link-preserving transformations of its lattice," and on his study of logic "with 4 and 16 truth-classes" (or, truth-values). [In this connection see A. A. Grau, Bull. Amer. Math. Soc. **53**, 567–572 (1947); these Rev. **9**, 3, and the paper of G. Birkhoff and S. A. Kiss reviewed below.] A large portion of the book is devoted to the exposition of those parts of modern algebra and symbolic logic which are necessary for an understanding of the above questions. The book does not purport to give a full account of the results of interaction, in recent years, between logic and lattice-theory; it is confined, instead, to a presentation of those aspects of the subject that have appealed to the author. For instance, there is no reference either to the ideal-theory of the distributive lattice and its connection with Tarski's theory of deductive systems, or to the theory of quotients in a distributive lattice and its connection with Brouwerian logic.

In estimating the value of his investigation, the author advances a conjecture that the physical and biological worlds require 4 and 16-class logics. He expects that "lattice theory will play a decisive part in the explanation of the quantum-phenomena of physics." He adduces, however, no evidence in support of his beliefs, nor does he give extensive illustrations, from other fields, of the uses of higher class logic. He contents himself with the statement that "4-class mathematics which would yield the desired examples has still to be invented by discarding the limitation which the exclusive use of 2-class logic has so far imposed on mathematical thought."

Finally, the reviewer cannot help remarking that the book contains a number of digressive comments of a strictly non-mathematical nature. *K. Chandrasekharan.*

Birkhoff, Garrett, and Kiss, S. A. A ternary operation in distributive lattices. Bull. Amer. Math. Soc. **53**, 749–752 (1947).

Étude, dans un treillis distributif, de l'opération $(a,b,c)=(a\cap b)\cup(b\cap c)\cup(c\cap a)=(a\cup b)\cap(b\cup c)\cap(c\cup a)$.

Ici $(a, x, b)=x$ équivaut à $a\cap b\geqq x\geqq a\cup b$. Les isométries respectent l'opération (a, b, c). Dans le cas de dimension finie, elles coïncident avec lès automorphismes respectant (a, b, c). Caractérisation des réseaux distributifs par l'opération (a, b, c) et comparaison avec d'autres opérations ternaires. *J. Kuntzmann* (Grenoble).

Fuchs, Ladislas. On quasi-primary ideals. Acta Univ. Szeged. Sect. Sci. Math. **11**, 174–183 (1947).

Let R be an integral domain with identity element. An ideal Q is defined to be quasi-primary (q. p.) if $ab\epsilon Q$ implies that some power of a or of b is in Q; Q is q. p. if and only if its radical is prime. If R is Noetherian, then every ideal is the intersection of a finite set of q. p. ideals such that no proper subset has q. p. intersection. The numbers of q. p. components and their radicals are uniquely determined. The connection between this decomposition and those given by E. Noether [Math. Ann. **83**, 24–66 (1921)] is discussed. *I. S. Cohen* (Philadelphia, Pa.).

✱Terpstra, Fedde Jan. Over Zekere Ongemengde Idealen. [On Some Unmixed Ideals]. Thesis, University of Amsterdam, 1946. viii+65 pp.

In a paper on the representation of biquadratic forms as sums of squares [Math. Ann. **116**, 166–180 (1938)] a theorem of Lasker was used, which states that a polynomial ideal $(f_1, \cdots, f_r)$ is unmixed of rank r if its rank is greater than or equal to r, where rank means the number of variables minus the dimension of the ideal. However, in the case needed for the application, the condition that the rank should be greater than or equal to r was not fulfilled, since f_1, f_2, f_3 were forms in x_1, x_2 and in y_1, y_2, y_3, having the trivial zero variety $x_1=x_2=0$ of dimension 3. Thus, r was 3, but the rank of (f_1, f_2, f_3) was only 2. Now, in order to save the argument, a theory of ideals in the ring of "bipolynomials" is developed in this thesis. Bipolynomials are defined as polynomials in the μm products $x_1y_1, \cdots, x_\mu y_m$. The variety of an ideal in this ring is defined as the set of points $(c_{11}, \cdots, c_{\mu m})$ in affine space, for which a decomposition $c_{ik}=a_ib_k$ is possible, so that all polynomials of the ideal become zero for $x_i=a_i$, $y_k=b_k$. The dimension of an ideal is defined by means of a maximal set, consisting of some x_i and some y_k, such that no bipolynomial in these x_i and y_k is in the ideal except zero. The number of these x_i and y_k is the dimension d of A, and the rank is $\mu+m-1-d$. The main theorem now runs: an ideal $(f_1, \cdots, f_r)$ in the ring of bipolynomials, of rank not less than r, is unmixed. *B. L. van der Waerden* (Baltimore, Md.).

McCoy, Neal H. Subdirect sums of rings. Bull. Amer. Math. Soc. **53**, 856–877 (1947).

Let $A=\{\alpha\}$ be any set and suppose that to each $\alpha\epsilon A$ there is associated a ring S_α. The set S of all functions s defined on A such that $s(\alpha)\epsilon S_\alpha$ is a ring called the direct sum of the rings S_α $(\alpha\epsilon A)$. Let T be a subring of S. For each $\alpha\epsilon A$ the mapping $\sigma_\alpha: t\to t(\alpha)$ is a homomorphism of T into S_α. We call T a subdirect sum of the rings S_α if each mapping $\sigma(\alpha)$ is onto, i.e. if $\sigma(\alpha)T=S_\alpha$, $\alpha\epsilon A$; the S_α are called the components of T. An isomorphism between a ring R and a subdirect sum T is called a representation of R as a subdirect sum. The present paper is a summary of some of the more important known results concerning subdirect sums of rings, and contains a fairly extensive bibliography. Proofs are given for many of the results and references supplied for others. Topics treated include general conditions for representations of rings as subdirect sums, with particular attention to commutative rings; subdirectly irreducible rings; various definitions for the radical of a ring; special subdirect sums; discrete direct sums. *R. M. Thrall* (Ann Arbor, Mich.).

Dieudonné, Jean. Sur les corps ordonnables. Bol. Soc. Mat. São Paulo **1**, 69–75 (1946).

Let a field K be called A-real if no element of $A\subset K$ is a sum of squares, A-real-closed if it is A-real but no proper algebraic extension is. The case where A consists solely of the element -1 is that of Artin-Schreier. The author shows that the main results can be generalized with appropriate modifications. In particular, suitable A-real-closed fields furnish examples of fields which are formally real, have no extensions of odd degree, and in which every sum of squares is a square, yet which are not real closed. [Reviewer's remark: moreover, any such field is A-real-closed if one takes A to be the set of nonsquares]. *I. Kaplansky*.

✱Pickert, Günter. Veränderung des Unvollkommenheitsgrades bei unendlichen, rein-inseparablen Erweiterungen. Ber. Math.-Tagung Tübingen 1946, pp. 118–120 (1947).

Let L be an infinite algebraic extension of a field K of characteristic p and $d(K)$ the degree of imperfection of K [Teichmüller, Deutsche Math. **1**, 362–388 (1936); Pickert, Math. Ann. **116**, 217–280 (1938), p. 220]. The following results are stated: (i) $d(L)\leqq d(K)$; (ii) if L can be obtained from K by adjoining all p^eth roots of the elements of some set $M\subset K$, and if n is the smallest cardinal number of such a set, then $d(L)\leqq d(K)-n$; equality holds for n finite or denumerable; (iii) (the other extreme case) if for each a in K there is a maximum e such that L contains the p^eth root of a, then $d(K)<d(L)$ is still possible, by use of an example of the reviewer [Duke Math. J. **5**, 372–393 (1939), pp. 389–392]. *S. Mac Lane* (Zurich).

Albert, A. A. A structure theory for Jordan algebras. Ann. of Math. (2) **48**, 546–567 (1947).

In a recent paper, the author developed a theory of Jordan algebras of linear transformations [Trans. Amer. Math. Soc. **59**, 524–555 (1946); these Rev. **8**, 63]. He now defines a Jordan algebra abstractly as a nonassociative algebra A over a field F such that the commutative law and the relation $a^2(ua)=(a^2u)a$ for any two elements a, u of A are satisfied. It is assumed that the characteristic of F is different from two. The polynomials of an element x of A with coefficients in F always form an associative algebra. Again, the concepts of solvability, nilpotency and strong nilpotency are equivalent. If every element of A is nilpotent, then A is solvable. Assuming from now on that F is a nonmodular field, the author shows that the radical of A can be characterized by a trace condition and that the same results hold for the radical of an abstract Jordan algebra as had been obtained in the former paper in the special case of Jordan algebras of linear transformations. The new proofs, however, are quite different. A Jordan algebra is semisimple if its radical vanishes. Every semisimple Jordan algebra is a direct sum of simple Jordan algebras. The residue class algebra of an arbitrary Jordan algebra modulo its radical is semisimple. The next problem is, of course, that of determining all simple algebras. The principal result is that every simple Jordan algebra over a nonmodular field is either isomorphic to a Jordan algebra of linear transformations or is an algebra of a very special type of order 27 over its center. *R. Brauer* (Toronto, Ont.).

NUMBER THEORY

Venkatachalam Iyer, R. Prime numbers whose reciprocals have the maximum recurring period. J. Univ. Bombay (N.S.) **15**, part 5, Sect. A., 9–10 (1947).

The author states the following conjecture without proof. Let $p=2m+1$ be a prime, where $2m$ is not a power of 10. If $10^m\equiv-1 \pmod p$, then $p-1$ is the length of the period in the decimal of the reciprocal of p. This conjecture is wrong. Consider, for instance, the prime $p=9901$. We have $10^6\equiv-1$, hence $10^{4950}\equiv-1 \pmod p$. But the length of the period is 12 since $10^{12}\equiv1 \pmod p$. *A. Brauer.*

✱Kanold, Hans Joachim. Kreisteilungspolynome und ungerade vollkommene Zahlen. Ber. Math.-Tagung Tübingen 1946, pp. 84–87 (1947).

Extending his results of earlier papers [J. Reine Angew. Math. **183**, 98–109 (1941); **184**, 116–123 (1942); these Rev. **3**, 268; **5**, 33] the author states results on the prime divisors of cyclotomic polynomials and criteria for the nonexistence of odd perfect numbers of certain types without proofs. The following results may be mentioned.

Let $F_m(x)$ be the mth cyclotomic polynomial, where $m=p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}\geqq3$ and $p_1<p_2<\cdots<p_k$. Then all the prime divisors of $y^{\varphi(m)}F_m(x/y)$ have the form $mz+1$ if $p_k\not\equiv1 \pmod{p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_{k-1}^{\alpha_{k-1}}}$. If $p_k\equiv1 \pmod{p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_{k-1}^{\alpha_{k-1}}}$, then p_k may be a divisor, but not p_k^2.

For every $m>6$ there exists a prime of the form $mz+1$ which is less than $5^{\varphi(m)/2}$.

Let p and q be primes. The Diophantine equation $1+q^a+q^{2a}+\cdots+q^{(p-1)a}=b^p$ has solutions only for $p=q=2$, $a=b=3$ and for $a=1$, $b=p=2$, $q=3$. The Diophantine equation $1+q^n+q^{2n}=b^n$ has no solution for $n>1$. [For $a=1$, this was proved by the reviewer, Bull. Amer. Math. Soc. **49**, 712–718 (1943); these Rev. **5**, 90.]

An odd number $n=p^\alpha q_1^{2\beta}q_2^2q_3^2\cdots q_r^2$ is not perfect in the following cases: (1) $2\beta<10$; (2) $n\not\equiv0 \pmod 3$ and $\alpha=5$; (3) $q_1=3$ and $2\beta=10$. If $q_1=3$, then n could be perfect only if $n>10^{36}$, $\alpha+2\beta+2(r-1)\geqq37$ and $r\geqq9$. *A. Brauer.*

Pérez-Cacho, Laureano. Fermat's last theorem and the Mersenne numbers. Revista Acad. Ci. Madrid **40**, 39–57 (1946). (Spanish)

This paper consists of two unrelated parts, each dealing with one of the two subjects of the title. In the first part various theorems equivalent to Fermat's last theorem are obtained in an elementary manner. The fundamental result is that the equation $a^{2n-1}+b^{2n-1}=c^{2n-1}$ has a solution in integers with $abc\neq0$ if and only if it is possible to find rational numbers x, y so that $(xy)^n=x+y$, $xy\neq0$. Here n is a positive integer but $2n-1$ need not be a prime. It follows at once that Fermat's theorem is true for this exponent if and only if z^2-a^nz+a is irreducible in the field of rational numbers for all rational a.

In the second part it is shown that if $2^n-1=BC$, then $B+C\equiv2^n \pmod{n^2}$ and $B+C\equiv0 \pmod{16}$. The possibility of a factorization of the form $2^n-1=x(4x-3)$, where x is an integer, is studied and is easily seen to be possible if and only if $2^{n+4}-7=h^2$, where h is an integer and $h\equiv5 \pmod 8$. For $n=11$ this leads to the known factorization $2^{11}-1=23\cdot89$ but no similar factorizations for larger n have been found. *H. W. Brinkmann* (Swarthmore, Pa.).

Lehmer, D. H. The Tarry-Escott problem. Scripta Math. **13**, 37–41 (1947).

Let $b>1$ and $\mu_1, \cdots, \mu_n>0$ be given. Let S_r denote the set of all the numbers $S=\mu_1a_1+\cdots+\mu_na_n$ for which $a_1+\cdots+a_n\equiv r \pmod b$, where every a_i ranges independently over all the integers $0, 1, \cdots, b-1$. Then the sums of the numbers of each set $S_0, S_1, \cdots, S_{b-1}$ are equal and so are the sums of the second powers, the sums of the third powers, $\cdots$, the sums of the $(n-1)$th powers. They constitute a chain of solutions of the Tarry-Escott problem of degree n (not $n+1$). In the proof of the theorem the author uses the function

$$F(x)=\prod_{\nu=1}^{n}(1+\epsilon e^{\mu_\nu x}+\epsilon^2e^{2\mu_\nu x}+\cdots+\epsilon^{b-1}e^{(b-1)\mu_\nu x});$$

$\epsilon\neq1$ is a bth root of unity. *N. G. W. H. Beeger.*

Bucher, J. Neues über die Pell'sche Gleichung. Mitt. Naturforsch. Ges. Luzern **14**, 1–18 (1943).

Let p, q be primes of the form $4n+1$, h the class number of the field $R(\sqrt{(pq)})$, and t, u the least positive integral solution of $t^2-pqu^2=4$. Using Dirichlet's results on the norm of the fundamental unit, the author proves that, if h is divisible by 4, then $\left(\frac{p}{q}\right)_4\equiv(\frac{1}{2}t)^{h/4} \pmod p$, where $\left(\frac{p}{q}\right)_4$ is the biquadratic character of p modulo q. Define $M=\prod(4\sin^2 x\pi/p-4\sin^2 y\pi/q)$, where the product ranges over all positive $x<p/2$ and positive $y<q/2$ for which $\left(\frac{x}{p}\right)=\left(\frac{y}{q}\right)=1$, and define λ by $\lambda M=|M|$. Let ϵ designate the conjugate of the fundamental unit in $R(\sqrt{p})$, define e by $e=-\epsilon\sqrt{p}$, and let $q=\alpha\alpha'$ be the factorization of q in $R(\sqrt{p})$. If h is divisible by 8, say $h=8h_1$, then $\lambda\left(\frac{e}{\alpha}\right)_4\equiv(\frac{1}{2}t)^{h_1} \pmod q$, and $\{(t-u\sqrt{(pq)})/2\}^{h_1}=|M|$. Finally, the coefficients of u in $\prod(u-4\sin^2 x\pi/p)$, the range of x in this product being the same as in M above, are integers in $R(\sqrt{p})$, and divisible by $\sqrt{p}$. Corresponding results are obtained throughout the paper for the case $q=2$. *I. Niven* (Eugene, Ore.).

Givens, Wallace. Parametric solution of linear homogeneous Diophantine equations. Bull. Amer. Math. Soc. **53**, 780–783 (1947).

The following stronger form of a result of L. W. Griffiths [Bull. Amer. Math. Soc. **52**, 734–736 (1946); these Rev. **8**, 6] is proved. The solutions of the Diophantine system $\sum_{i=1}^{n}a_{\alpha i}x_i=0$ $(\alpha=1, 2, \cdots, r)$ of rank r admit the representation $x_i=\Delta_i/a$ in terms of integer parameters $p_{\alpha i}$ $(\alpha=r+1, \cdots, n-1; i=1, \cdots, n)$; a is the g.c.d. of the r-rowed minors of the matrix $A=\|a_{\alpha i}\|$ and Δ_i is the ith $(n-1)$-rowed minor of the matrix $\|q_{\alpha i}\|$ $(q_{\alpha i}=a_{\alpha i}$ for $1\leqq\alpha\leqq r$, $q_{\alpha i}=p_{\alpha i}$ for $r<\alpha\leqq n-1)$. *N. G. de Bruijn* (Delft).

Bambah, R. P. Ramanujan's function $\tau(n)$—a congruence property. Bull. Amer. Math. Soc. **53**, 764–765 (1947).

A short proof of the congruence $\tau(n)\equiv n\sigma_3(n) \pmod 7$. *D. H. Lehmer* (Berkeley, Calif.).

Bambah, R. P., Chowla, S., and Gupta, H. A congruence property of Ramanujan's function $\tau(n)$. Bull. Amer. Math. Soc. **53**, 766–767 (1947).

A short proof of the congruence $\tau(n)\equiv\sigma(n)$ or $0 \pmod 8$ according as n is odd or even. *D. H. Lehmer.*

Bambah, R. P., and Chowla, S. A new congruence property of Ramanujan's function $\tau(n)$. Bull. Amer. Math. Soc. 53, 768–769 (1947).

A short proof of the congruence $\tau(n) \equiv n^2\sigma(n)$ (mod 9).
D. H. Lehmer (Berkeley, Calif.).

Ramanathan, K. G. Congruence properties of $\sigma_a(N)$. Proc. Indian Acad. Sci., Sect. A. 25, 314–321 (1947).

The author gives his own proofs of results noted by him previously and proved otherwise by H. Gupta [Math. Student 13, 25–29 (1945); these Rev. 7, 273; see also Ramanathan, Math. Student 11, 33–35 (1943); these Rev. 6, 37]. *D. H. Lehmer* (Berkeley, Calif.).

Davenport, H., and Heilbronn, H. On the minimum of a bilinear form. Quart. J. Math., Oxford Ser. 18, 107–121 (1947).

The authors study the minimum $M(B)$ of the absolute value of $B(x, y, z, t) = (\alpha x + \beta y)(\gamma z + \delta t)$, where $\alpha, \beta, \gamma, \delta$ are real constants and x, y, z, t are real variables subject to $xt - yz = \pm 1$, whereas they suppose $\Delta = \alpha\delta - \beta\gamma \neq 0$; α/β and γ/δ irrational. For all forms of this kind they prove

$$M(B) \leqq \frac{3-\sqrt{5}}{2\sqrt{5}} |\Delta|.$$

For all forms for which this minimum is not attained, they find $M(B) \leqq \frac{1}{4}(2-\sqrt{2})|\Delta|$ and for all forms where this second minimum is not attained, they find $M(B) \leqq \frac{1}{2}(\sqrt{2}-1)|\Delta|$. Whereas in the first two cases only the forms which are equivalent to a multiple of one particular form attain the minimum named above, in the third case the following unexpected theorem holds. For any $\delta > 0$ there exists a set of forms, no one of which is equivalent to a multiple of another, for which $M(B) > \frac{1}{2}((\sqrt{2}-1)-\delta)|\Delta|$, and the set has the cardinal number of the continuum. This theorem shows a fundamental difference from the well-known similar classical results of Markoff on binary quadratic forms [Math. Ann. 15, 381–407 (1879); 17, 379–400 (1880)].
J. F. Koksma (Amsterdam).

Davenport, H. Non-homogeneous binary quadratic forms. III. Nederl. Akad. Wetensch., Proc. 50, 484–491 = Indagationes Math. 9, 290–297 (1947).

In this paper the author treats the case

$$f(x, y) = 5x^2 - 11xy - 5y^2.$$

Parts I and II appeared in the same Proc. 49, 815–821 (1946); 50, 378–389 (1947); these Rev. 8, 444, 565.
J. F. Koksma (Amsterdam).

Oppenheim, A. A positive definite quadratic form as the sum of two positive definite quadratic forms. I. J. London Math. Soc. 21 (1946), 252–257 (1947).

Oppenheim, A. A positive definite quadratic form as the sum of two positive definite quadratic forms. II. J. London Math. Soc. 21 (1946), 257–264 (1947).

If ϕ is a positive definite integral quadratic form in $x_1, \cdots, x_n$, the problem is to determine a decomposition $\phi = \phi_1 + \phi_2$, where ϕ_1, ϕ_2 are also positive definite integral quadratic forms. Let $L(\phi)$ denote the minimum of ϕ for all integers $x_1, \cdots, x_n$, the set $0, \cdots, 0$ excepted. In (II) the author proves the following results. (1) There exists a least integer $K_n \geqq 2$ such that all positive definite classical integral forms ϕ with $L(\phi) \geqq K_n$ are decomposable. There exist indecomposable forms ϕ with $L(\phi) < K_n$. (2) There exists a corresponding integer K_n^* for the case of positive definite integral forms (not necessarily classical). In (I) it is proved that (a) $K_2 = 3$, $K_3 = 3$, (b) $K_2^* = 2$, $K_3^* = 2$. [Actually the existence of K_n and K_n^* follows immediately from a theorem of Bieberbach and Schur, S.-B. Preuss. Akad. Wiss., Phys.-Math. Kl. 1928, 510–535.] *H. S. A. Potter* (Aberdeen).

Mahler, K. On the generating functions of integers with a missing digit. K'o Hsüeh (Science) 29, 265–267 (1947). (Chinese)

Let N be the set of positive integers n which contain no digit equal to zero. The author proposed the problem of whether or not $\sigma = \sum 1/n$ for $n \epsilon N$ is a transcendental number. In this paper the author considers an analogous problem for $f(z) = \sum z^n$, $n \epsilon N$. He studies the analytic behavior of the function $f(z)$ and proves that $f(z)$ is a transcendental number if z is algebraic and $0 < |z| < 1$. [Cf. Math. Ann. 101, 342–366 (1929).] *L.-K. Hua* (Princeton, N. J.).

***Lock, Didericus Jacobus. Metrisch-Diophantische Onderzoekingen in $K(P)$ en $K^{(n)}(P)$. [Metric-Diophantine Investigations in $K(P)$ and $K^{(n)}(P)$].** Thesis, Free University of Amsterdam, 1947. vii+100 pp.

The aim of this thesis is to develop metrical theorems on Diophantine approximation in the field of P-adic numbers $K(P)$ and in metricised product fields $K^{(n)}(P)$. In close agreement with the thesis of H. Turkstra [Free University of Amsterdam, 1936], Lebesgue and Hausdorff measures are developed. In contradistinction to Turkstra, the author starts from the axioms of Carathéodory [chapter II]. Chapter III gives an outline of the Diophantine theory; chapter IV contains the proof of the p-adic analogue of Khintchine's theorem on the simultaneous approximation of the number system $\theta, \theta^2, \cdots, \theta^n$. This proof comes from Turkstra, who mentioned this result in his thesis. Chapter V contains the complete transcription of Khintchine's theorem on the approximation of the system $(\theta_1, \theta_2, \cdots, \theta_n)$, whereas chapter VI transcribes the refinement of this theorem in the case $n = 1$ by Jarník with use of Hausdorff measure. The final chapter VI contains the p-adic analogue of investigations of Mahler and the reviewer on the classification of transcendental numbers. Some results on the extension fields of $K(P)$, familiar in modern algebra, which had to be used here, have been systematically included in the introductory chapter I. For literature see the reviewer's Diophantische Approximationen [Ergebnisse der Math., v. 4, no. 4, Springer, Berlin, 1936], and Monatsh. Math. Phys. 48, 176–189 (1939); these Rev. 1, 137. *J. F. Koksma.*

Errera, A. Sur le théorème de MM. Khintchine et Mann. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 32 (1946), 300–306 (1947).

Expository paper on the density of sums of sets of integers.
H. B. Mann (Columbus, Ohio).

van der Corput, J. G. On sets of integers. III. Nederl. Akad. Wetensch., Proc. 50, 429–435 = Indagationes Math. 9, 257–263 (1947).

[For parts I and II cf. the same Proc. 50, 252–261, 340–350 = Indagationes Math. 9, 159–168, 198–208 (1947); these Rev. 8, 566.] Let A_{hk} ($h = 1, \cdots, c$; $k = 1, \cdots, n_h$) be sets of nonnegative integers containing the number 0 and satisfying the inequalities $\sum_j A_{hj}(m) \geqq \gamma_h m$, $m = 1, \cdots, g$, $\sum_i \gamma_i \leqq 1$. The author forms sums T of the form $A_{h_1k_1} + A_{h_2k_2} + \cdots$ such that every pair of indices occurs at most once. He then considers a sum $\sum_1 T(m)$ which for every h is symmetrical

with respect to k. It is proved that $\sum_1 T(m) \geqq m\sum_1 \tau(T)$ for $m=1, \cdots, g$, where $\tau(T)=\lambda_1\gamma_1/n_1+\cdots+\lambda_e\gamma_e/n_e$ and λ_h denotes for each h the number of terms A_{hk} occurring in T. Two previous results of F. J. Dyson are corollaries of this result. Let, further, the positive numbers $f(m)$ satisfy the inequalities $f(m+1)\geqq f(m)$, $f^2(m+1)\geqq f(m)f(m+2)$. Let $A_1, \cdots, A_n$ be sets of nonnegative integers containing the number 0 and satisfying the inequalities $A_1(m)+\cdots+A_n(m)\geqq\gamma\sum_1^m f(h)$, $\gamma\leqq 1$. The author proves that $(A_1+\cdots+A_n)(m)\geqq\gamma\sum_1^m f(h)$.
H. B. Mann (Columbus, Ohio).

Mirsky, L. On the number of representations of an integer as the sum of three r-free integers. Proc. Cambridge Philos. Soc. **43**, 433–441 (1947).

The error term in Evelyn and Linfoot's formula for the number of representations of a number n as the sum of three numbers not divisible by rth powers [Math. Z. **34**, 637–644 (1932), p. 638, (1.3), case $s=3$] is reduced from $O(n^{1+2/(r+1)+\epsilon})$ to $O(n^{1+8/(3r+1)+\epsilon})$.
T. Estermann (London).

Mirsky, L. Note on an asymptotic formula connected with r-free integers. Quart. J. Math., Oxford Ser. **18**, 178–182 (1947).

Let $k_1, \cdots, k_s$ be any integers. Denote by $F(x)$ the number of integers $n\leqq x$ such that all the integers $n+k_1$, $n+k_2$, $\cdots$, $n+k_s$ are r-power-free (i.e., not divisible by the rth power of any integer). The author proves in an elementary way that $F(x)=cx+O(x^{2/(r+1)+\epsilon})$. The constant c is given by a complicated expression.
P. Erdős (Syracuse, N. Y.).

Guinand, A. P. Discontinuous limits and Fourier-Stieltjes integrals. Quart. J. Math., Oxford Ser. **18**, 72–84 (1947).

In the first part of the paper it is shown that, if $r(n)$ denotes the number of ways of expressing n as the sum of two squares, then

$$\lim_{T\to\infty} T^{-1}\sum_{n<T^2} n^{-\frac{1}{4}}r(n)\cos((2\pi n)^{\frac{1}{2}}y-\pi/4)$$

exists and is either $k^{-\frac{1}{4}}r(k)$ or 0 according as y is or is not of the form $(2\pi k)^{\frac{1}{2}}$. The proof depends on asymptotic formulae for the Bessel functions J_0, J_1 and on known summation formulae involving $r(n)$ and these functions. The author gives examples of other limit relations involving number-theoretic functions and Bessel functions which can be proved in an analogous manner. [It should be pointed out that the above result concerning $r(n)$ can be obtained from an integration by parts of statement (i) of Wintner, Amer. J. Math. **63**, 619–627 (1941), p. 620; these Rev. **2**, 350.]

In the second part of the paper, a general class of similar limit relations obtainable from Fourier-Stieltjes transforms is discussed, viz., if $f(x)$ and its transform $g(x)$, where

$$\int_0^y g(x)dx=(2/\pi)^{\frac{1}{2}}\int_0^\infty x^{-2}(1-\cos xy)df(x),$$

satisfy certain conditions (concerning local smoothness and behavior at $x=0$ and $x=\infty$), then

$$\lim_{T\to\infty}(2\pi)^{\frac{1}{2}}T^{-1}\int_0^T \cos xy\, df(x)$$

exists and equals $g(y+0)-g(y-0)$.
P. Hartman.

Turán, P. On a theorem of Littlewood. J. London Math. Soc. **21** (1946), 268–275 (1947).

The basic result is (essentially): If $G(y)=\sum_{j=1}^n b_j z_j^y$, where $|z_j|\geqq 1$ $(j=1,\cdots,n)$, then every interval $m-n\leqq\nu\leqq m$ $(m>n)$ contains an integer ν such that $|G(\nu)|\geqq|G(0)|(n/Am)^n$, where A is a numerical constant (for which $A=33$ is a possible value). A simple algebraical proof is given. An easy deduction is that, if $\xi>1$, $\lambda_j>0$, then

$$(*)\qquad \max_{\xi-1\leqq x\leqq\xi}\left|b_0+\sum_{j=1}^N b_j\cos\lambda_j x\right|\geqq(A\xi)^{-2N-1}\left|\sum_{j=0}^N b_j\right|.$$

The factor $(A\xi)^{-2N-1}$ cannot be replaced by anything greater than $(\pi/2\xi)^{2N}$. This is an extension and simplification of Littlewood's "inequality for a sum of cosines" [see the same J. **12**, 217–221 (1937)]. It is pointed out (since there was a slip in Littlewood's statement of what he actually proved) that there is no one-sided (algebraic) version of (*), even if the b_j are positive. The object of these and similar inequalities is to show that certain sums arising in the analytical theory of primes do not become too small through mutual interference of their terms. [For the author's application to the "quasi-Riemannian hypothesis," briefly noted here, see the following review.]
A. E. Ingham.

Turan, Paul. On Riemann's hypothesis. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **11**, 197–262 (1947). (English. Russian summary)

The following statements are shown (in theorem II) to be equivalent. (i) There is a $\vartheta<1$ such that $\zeta(s)$ has at most a finite number of zeros in the half-plane $\sigma>\vartheta$; (ii) there are numbers α, β $(\alpha\geqq 2, 0<\beta\leqq 1)$ such that

$$\left|\sum_{N'\leqq p\leqq N''} e^{it\log p}\right|<ANe^{22(\log\log N)^2}/|t|^\beta$$

(p prime) whenever $a\leqq|t|^\alpha\leqq\frac{1}{2}N\leqq N'<N''\leqq N$, where A and a are appropriate constants. The assumption (i) (with a "numerically given" ϑ) is called, after Kalmár, the "quasi-Riemannian hypothesis." It implies, of course, (i′): there is a $\vartheta'<1$ such that $\zeta(s)$ has no zeros in $\sigma>\vartheta'$. But ϑ seems to be more directly related than ϑ' to α and β (though no final form of connection is attempted in either case). The separate implications (i)→(ii) and (ii)→(i) are refined in various ways. Of special interest are theorems VI and VII, which relate the assertion (ii) for finite ranges of t to the nonexistence of zeros in half-strips $\sigma>\vartheta$, $t_1<t<t_2$, thus localising to some extent the connection between primes and zeros.

The proof that (ii)→(i) (the more difficult implication) is based on an identity equivalent to

$$\frac{1}{(\nu-1)!}\sum_{n>\xi}\frac{\Lambda(n)}{n^s}\log^{\nu-1} n/\xi=\frac{\xi^{1-s}}{(s-1)^\nu}-\sum_\rho\frac{\xi^{\rho-s}}{(s-\rho)^\nu}-\sum_{n=1}^\infty\frac{\xi^{-2n-s}}{(s+2n)^\nu},$$

$(\sigma>1, \xi\geqq 1, \nu=2, 3, \cdots)$. The idea is to estimate the left hand side by (ii), and to show that, if $\zeta(s)$ had a zero ρ^* too near $\sigma=1$, the contribution of the term $\rho=\rho^*$ to the right hand side could be made too large by taking s near ρ^* (subject to $\sigma>1$), ν large (of order $\log t$), and $\xi=e^\nu$. The danger of interference from other terms ρ near s is met by a series of algebraical lemmas (of independent interest) designed to show that ν can be so chosen (within the permitted range) that this interference is not too great. A typical example is lemma XII: if $\max(|z_1|, \cdots, |z_n|)=1$, $m\geqq 28n$, then

$$\max_{m\leqq\nu\leqq m+n}|z_1^\nu+z_2^\nu+\cdots+z_n^\nu|>n^{-2}(e^{-28}n/m)^n.$$

The condition (ii) is of a novel type among necessary and sufficient conditions for (i) or (i′) in that the sum involved can in principle be estimated by known methods independ-

ent of the theory of $\zeta(s)$. An appendix is devoted to the study by Vinogradov's method of various modifications of this sum. If, for example, $\log p$ is replaced by $(\log p)^c$ $(\frac{1}{2}\leqq c\leqq 2)$, the direct attack fails when $c=1$, but the author shows how we may argue from a smaller c to a larger c' (theorem X is the case $c=\frac{1}{2}$, $c'=1$), and hazards a conjecture for the case $c<1$ that would carry with it (ii) and therefore (i). *A. E. Ingham* (Cambridge, England).

Rédei, L. Bemerkung zu meiner Arbeit "Über die Gleichungen dritten und vierten Grades in endlichen Körpern." Acta Univ. Szeged. Sect. Sci. Math. 11, 184–190 (1947).

[The paper referred to in the title appeared in the same vol., 96–105 (1946); these Rev. 8, 138.] Let k denote the $GF(p^f)$, $p>3$, and put

$$\varphi_n(x)=\binom{n}{1}+\binom{n}{3}x+\binom{n}{5}x^2+\cdots.$$

For the cubic $f(x)=a_0x^3+a_1x^2+a_2x+a_3$, $a_i\epsilon k$, put $\Delta=-27f(a_1/3a_0)/a_0D$, where D is the discriminant of $f(x)$. The following criterion is proved. Let ν be the number of roots of $f(x)=0$ in k. Assume $\Delta\neq 0$. Then $\nu=1$ if and only if $\varphi_{p^f+1}(-3\Delta)=0$; $\nu=3$ if and only if $\varphi_r(\Delta)=0$, where $r=[(q+1)/3]$. Use is made of the following theorem. The discriminant of a polynomial in k without repeated factors is a square in k if and only if the number of irreducible factors of even degree is an even number. The author gives a simple proof of this theorem; it may be noted that for $t=1$, it reduces to a theorem of Stickelberger [Verh. 1. Int. Math. Kongr., Zürich, 1897, pp. 182–193]. *L. Carlitz.*

Cohen, Eckford. Sums of an even number of squares in $GF[p^n, x]$. Duke Math. J. 14, 251–267 (1947).

Let $\alpha_1, \cdots, \alpha_{2s}$ $(s\geqq 1)$ be given nonzero elements of $GF(p^n)$, p an odd prime, let m be a fixed integer such that $2s\geqq m\geqq 1$, and let $\beta=\alpha_1+\cdots+\alpha_m$. Let $\delta_s(M)$, $\gamma_s(M)$ and $\rho_s(M,\mu)$ be the divisor functions, for a polynomial M in $GF[p^n, x]$, previously defined by Carlitz and the author [same J. 14, 13–20 (1947); these Rev. 8, 503]. The omission of the superscript from $\delta_s^e(M)$, etc., signifies $e=1$. The main result of the present paper is as follows. If $\beta\neq 0$ and a given F in $GF[p^n, x]$ is primary (i.e., monic) of degree $2k$, or if $\beta=0$ and F is arbitrary of degree less than $2k$, then the equations $\beta F=\alpha_1X_1^2+\cdots+\alpha_{2s}X_{2s}^2$, or $F=\alpha_1X_1^2+\cdots+\alpha_{2s}X_{2s}^2$, respectively, have $\rho_{s-1}(F,\lambda)$ sets of solutions $X_1, \cdots, X_{2s}$ in $GF[p^n, x]$ such that $X_1, \cdots, X_m$ are primary of degree k and $X_{m+1}, \cdots, X_{2s}$ are arbitrary of degree less than k, where $\lambda=+1$ or -1 according as $(-1)^s\alpha_1\cdots\alpha_{2s}$ is or is not a square in $GF(p^n)$. Preliminary theorems involving the divisor functions are proved and corollaries giving expressions for numbers of solutions as polynomials in p are stated. The excluded case $m=0$ is not treated. *R. Hull.*

ANALYSIS

Jecklin, H., und Eisenring, M. Die elementaren Mittelwerte. Mitt. Verein. Schweiz. Versich.-Math. 47, 123–165 (1947).

A function $M=f(a_1, \cdots, a_n)$ of the positive variables a_k is called a mean if it is real-valued, continuous, symmetrical and such that $\min\{a_k\}\leqq M\leqq\max\{a_k\}$. The simplest algebraic means are the fundamental means $M(s_k)=\{s_k/\binom{n}{k}\}^{1/k}$ $(k=1, \cdots, n)$, s_k denoting the kth elementary symmetric function of the a's. Generalizing the classical inequality between the arithmetic mean $M(s_1)$ and the geometric mean $M(s_n)$, it is shown that $M(s_1)\geqq M(s_2)\geqq M(s_3)\geqq\cdots\geqq M(s_n)$. It is shown that the fractions

$$M_{tr}=\{M^t(s_t)/M^r(s_r)\}^{1/(t-r)}, \qquad t\neq r,$$

are also means and the following inequalities are established: $M(s_1)\geqq M_{21}\geqq M_{32}\geqq\cdots\geqq M_{n,n-1}$. The last of these means is the harmonic mean, and so this is a generalization of the classical inequality between the arithmetic and the harmonic means.

The fundamental means are used to build up various more general types of means. The most interesting are the means $M(k,h)=\{\sum a^{k+h}/\sum a^k\}^{1/h}$, where k and h are arbitrary real numbers, $h\neq 0$. One may define $M(k,0)$ as the limit of $M(k,h)$ for $h\to 0$. In particular, $M(0,0)$ is the geometrical mean. It is shown that

$$M(k,h)\leqq M(k+1,h),\quad M(k,h)\leqq M(k,h+1),$$
$$M(k,h+1)\leqq M(k+1,h).$$

When one of the variables k, h is fixed and the other tends to ∞ $(-\infty)$, then $M(k,h)$ tends to $\max\{a\}$ $(\min\{a\})$. The radial limit $\lim_{r\to\infty} M(r\cos\varphi, r\sin\varphi)$ is equal to $\max\{a\}$ for $-\pi/4\leqq\varphi\leqq\pi/2$ and to $\min\{a\}$ for $3\pi/4\leqq\varphi\leqq 3\pi/2$, while in the remaining two sectors it has an intermediate value, depending only on $\max\{a\}$, $\min\{a\}$ and φ.

The means $M(k,h)$ are special cases of the following ones:

$$M(k,h;r,q)=\left\{{}_{r+q}p_{k+h}\cdot\binom{n}{r}\Big/{}_rp_k\cdot\binom{n}{n+q}\right\}^{1/(k q+h(r+q))},$$

where ${}_rp_k$ denotes the symmetric function $(a_1\cdots a_r)^k+\cdots$, the sum being extended over all combinations of the class r. Inequalities between these means and their asymptotic behaviour are also studied. *B. de Sz. Nagy* (Szeged).

Batty, Joyce S. Sets of non-integral functional powers. Quart. J. Math., Oxford Ser. 18, 85–96 (1947).

The author continues investigations by Batty and Walker [same J. 17, 146–152 (1946); these Rev. 8, 199; referred to as F.P.] and Walker [same J. 17, 65–82, 83–92 (1946); these Rev. 8, 19; referred to as C.F.(I) and C.F.(II)]. We refer to the reviews of these papers, especially the first, for the definition of terms unexplained in the present review. Let $P=P(f,\Lambda)$ be a power set of functions, where $f=f(x)$ $(a\leqq x\leqq b)$ is the "base function" (greater than the identity function), and Λ the set of all "indices" λ such that P consists of all "powers" f^λ. The paper is concerned with the case that Λ is everywhere dense. The always existing "limit function" of $P(f,\Lambda)$ is defined as $L(x)=\lim_{\lambda\to 0+}f^\lambda(x)$. Since this limit turns out to be uniform if and only if $L(x)=x$, the power set P is called uniform if $L(x)=x$. It is proved that P is uniform if and only if P is "related" (cf. C.F.(I)). The author proceeds then to the investigation of nonuniform power sets P and proves the following facts concerning the limit function $L(x)$ of such P: (i) L is constant in each of a set of nonoverlapping closed intervals δ called the "basic intervals"; (ii) if $y<z$ are the endpoints of such an interval δ then $L(x)=z$ in this δ; (iii) the set of basic intervals is everywhere dense in (a,b) and the complementary set T is nowhere dense; (iv) for $t\epsilon T$, $L(t)=t$; (v) $L(x)$ is continuous except at the left hand endpoints of the basic

intervals δ; (vi) $L(x)$ is independent of the choice of the base function f of P. Each function f^{λ} of P maps (a, b) on itself in such a way that a basic interval δ_0 is mapped on a basic interval δ_λ. The set $\{\delta_\lambda\}$ of intervals δ_λ thus obtained from δ_0 if λ takes all values of Λ thus turns out to be a subset of the basic intervals. If $\{\delta_\lambda\}$ is identical with the set of basic intervals, P is called a "total" power set. The above facts yield a nearly immediate proof of a conjecture stated in F.P., § 7: if Λ is not countable then P is uniform (or what is the same, related). The remainder of the paper is devoted to the construction of different types of nonuniform power sets. *E. H. Rothe* (Ann Arbor, Mich.).

Bradley, F. W., and Walker, A. G. Existence theorems for non-uniform power-sets. Quart. J. Math., Oxford Ser. **18**, 97–106 (1947).

A continuation of the investigations of the paper of the preceding review. The main result concerning the existence of nonuniform power sets is contained in the following theorem: given (i) a countable corpus Λ (i.e., a set containing with any two of its members their sum and difference) which is everywhere dense and includes unity; (ii) a strictly increasing function $f=f(x)$ in $a\leqq x\leqq b$ such that $f(x)>x$ for interior points and $f(x)=x$ at the endpoints; (iii) a step function $L=L(x)$ which is constant in each of a set of basic intervals in (a, b) and satisfies $Lf=fL$, then there exists a nonuniform power set having f as base function, Λ as set of indices (with respect to f) and L as limit function. The remaining part of the paper is devoted to the discussion of "partial" power sets, i.e., power sets which are not total [see the preceding review]. A condition is given under which a partial nonuniform power set can be augmented to a total power set. It is also proved that there are nonuniform partial power sets which cannot be augmented to become total. *E. H. Rothe.*

Loève, Michel. Remarques sur la majoration de certaines transformées. C. R. Acad. Sci. Paris **225**, 31–33 (1947).

The author points out that the recent extension by R. P. Boas [same C.R. **224**, 1683–1685 (1947); these Rev. **8**, 569] of a result of Y. Tagamlitzki [same C.R. **223**, 940–942 (1946); these Rev. **8**, 259] in its turn is a special case of comparison theorems for moment sequences, Laplace-Stieltjes and Fourier-Stieltjes transforms. Let $\alpha(t)$, $\beta(t)$ be of bounded variation and normalized in the interval considered and let $\beta(t)$ be nondecreasing. (I) Let $\{a_n\}_1^\infty$ and $\{b_n\}_1^\infty$ be sequences of real numbers. The condition $|\Delta^p a_n|\leqq(-1)^p\Delta^p b_n$ for all n and p, which forces $\{b_n\}$ to be a Hausdorff moment sequence, $b_n=\int_0^1 t^n d\beta(t)$, is necessary and sufficient in order that $\{a_n\}$ shall also be a moment sequence, $a_n=\int_0^1 t^n d\alpha(t)$ with $|\Delta\alpha(t)|\leqq\Delta\beta(t)$. (II) Let $\{a_n\}_{-\infty}^{\infty}$ and $\{b_n\}_{-\infty}^{\infty}$ be sequences of complex numbers with $\bar{a}_n=a_{-n}$, $\bar{b}_n=b_{-n}$. Let $\{\lambda_j\}$ be arbitrary complex numbers. The condition

$$\left|\sum_1^N \sum_1^N a_{i-j}\lambda_i\bar{\lambda}_j\right|\leqq\sum_1^N \sum_1^N b_{i-j}\lambda_i\bar{\lambda}_j$$

for all N and all λ's, which forces $\{b_n\}$ to be a trigonometric moment sequence, $b_n=\int_{-\pi}^{\pi} e^{int}d\beta(t)$, is necessary and sufficient in order that $a_n=\int_{-\pi}^{\pi} e^{int}d\alpha(t)$ with $|\Delta\alpha(t)|\leqq\Delta\beta(t)$. The comparison theorems for Laplace-Stieltjes and Fourier-Laplace transforms are based upon kernels of positive type and are entirely similar to the preceding results. In each case the inequality $|\Delta\alpha(t)|\leqq\Delta\beta(t)$ implies that intervals of constancy of $\beta(t)$ are also intervals of constancy of $\alpha(t)$, points of continuity of $\beta(t)$ are points of continuity of $\alpha(t)$, and $\alpha(t)$ is continuous (absolutely continuous) whenever $\beta(t)$ has the same property. *E. Hille* (New Haven, Conn.).

Theory of Sets, Theory of Functions of Real Variables

Luzin, N. N. On subsets of the series of natural numbers. Izvestiya Akad. Nauk SSSR. Ser. Mat. **11**, 403–410 (1947). (Russian)

This paper contains proofs for results announced previously [C. R. (Doklady) Acad. Sci. URSS (N.S.) **40**, 175–178 (1943); these Rev. **6**, 42], as well as extensions of these results, the statement of several unsolved problems, and general remarks. Let R denote the sequence of positive integers in their usual order; let $E\equiv 0$ for $E\subset R$ mean that E is a finite set; let $E\dashv F$ mean that $E\cap F'\equiv 0$; let A and B, subsets of R, be called orthogonal if $A\cap B\equiv 0$. A family $\mathfrak{M}$ of subsets of R is said to have a set H in R as a cover if $M\dashv H$ for all $M\epsilon\mathfrak{M}$. Two families $\mathfrak{M}$ and $\mathfrak{N}$ of subsets of R are said to be separated if they have disjoint covers and are said to be orthogonal if their elements are pairwise orthogonal. A family $\{E_1, E_2, \cdots, E_\alpha, \cdots\}$ $(\alpha<\alpha_0)$ of subsets of R is said to be increasing (decreasing) if $E_\alpha\dashv E_\beta$ $(E_\alpha\vdash E_\beta)$ for all $\alpha<\beta<\alpha_0$, and it is said to be strictly increasing (strictly decreasing) if there exists an ordinal number $\lambda<\alpha_0$ such that $\alpha>\beta>\lambda$ implies that $E_\alpha\not\equiv E_\beta$.

The following theorems are proved. (1) (du Bois-Reymond) If $\mathfrak{M}$ and $\mathfrak{N}$ are countable orthogonal families of subsets of R, then $\mathfrak{M}$ and $\mathfrak{N}$ are separated. (2) There exist orthogonal families $\mathfrak{M}_0$ and $\mathfrak{N}_0$ of subsets of R, both having cardinal number $\aleph_1$, which are not separated. (3) There exist families $\mathcal{E}=\{E_1, E_2, \cdots, E_\alpha, \cdots\}$ $(\alpha<\Omega)$ and $\mathfrak{F}=\{F_1, F_2, \cdots, F_\alpha, \cdots\}$ $(\alpha<\Omega)$ which are strictly increasing, orthogonal, and have the property that $\mathcal{E}$ and $\mathfrak{F}$ are not separated. (4) There exist families $\mathcal{E}$ and $\mathfrak{F}$ as described in (3) which are separated.

Several unsolved problems are proposed. (1) To exhibit or prove the nonexistence of a family $\mathcal{E}$ of cardinal number $\aleph_0$ and a family $\mathfrak{F}$ of cardinal number $\aleph_1$ of subsets of R such that $\mathcal{E}$ and $\mathfrak{F}$ are orthogonal but not separated. (2) To prove or disprove the existence of two families $\mathcal{E}=\{E_1, E_2, \cdots, E_\alpha, \cdots\}$ and $\mathfrak{F}=\{F_1, F_2, \cdots, F_\alpha, \cdots\}$ $(\alpha<\Omega)$ such that $\mathcal{E}$ and $\mathfrak{F}$ are strictly increasing and such that there exists exactly one set H (identifying sets which differ by a finite set) such that $H\vdash E_\alpha$ and $H\vdash F_\alpha$ for all $\alpha<\Omega$. (3) To prove or disprove the existence of a strictly decreasing sequence $\{E_1, E_2, \cdots, E_\alpha, \cdots\}$ $(\alpha<\Omega)$ such that for no infinite subset E of R does the relation $E_\alpha\vdash E$ obtain for all $\alpha<\Omega$. The author states that these problems, in his opinion, cannot be solved under the present points of view concerning the transfinite and the sequence of natural numbers. The paper concludes with some general remarks concerning the continuum problem and infinite sets. *E. Hewitt.*

Cuesta, N. Number of order types. Revista Mat. Hisp.-Amer. (4) **6**, 59–65 (1946). (Spanish)

This paper is concerned with the potency of various classes of order types. The first two theorems are as follows. (1) If m is a cardinal number greater than or equal to $2^{\aleph_0}$, the potency of the class of continuous order types of cardinal number m is 2^m. (2) If $m>\aleph_0$, the number of dense order types of cardinal number m in which no continuous order types can be imbedded is 2^m. The remaining three

theorems concern the potency of classes of order types containing particular sorts of discontinuities as classified by the author's method [same Revista (4) 3, 186–205, 242–268 (1943); these Rev. 5, 231]. *J. V. Wehausen.*

Liapounoff, A. Sur les ensembles projectifs, qui admettent des décompositions régulières. Rec. Math. [Mat. Sbornik] N.S. **20**(62), 179–196 (1947). (Russian. French summary)

It has been shown by Kondô [Proc. Imp. Acad. Tokyo **13**, 287–291 (1937)] that every projective set $\mathcal{E}\subset I_x$ of the second class, A_2, is a one-to-one and continuous image of an analytic complement, $E\subset I_{xy}$. Then E may be decomposed into $\aleph_1$ Borel sets in I_{xy}, $E=\sum E_\beta$. This induces a decomposition in I_x of $\mathcal{E}=\sum\mathcal{E}_\beta$, where the $\mathcal{E}_\beta$ are also Borel sets. The author is concerned with the subclass $\bar{A}_2$ of the class A_2 consisting of sets for which there exists a $\gamma<\Omega$ such that (a) $\sum_{\beta>\gamma}\mathcal{E}_\beta$ has the property of Baire in the restricted sense, and (b) for any completely additive set function $F(U)$ defined for all Borel sets in I_x, $F(\sum_{\beta<\gamma}\mathcal{E}_\beta)=F(\mathcal{E})$, $F(\sum_{\beta\geq\gamma}\mathcal{E}_\beta)=0$. The class of complements of sets in $\bar{A}_2$ is denoted by $C\bar{A}_2$, and the sets common to both classes by $\bar{B}_2$. The property of belonging to one of these classes is invariant under homeomorphism. (1) The class A is contained in the class $\bar{A}_2$. (2) The class $\bar{A}_2$ is invariant with respect to the operations of forming finite or denumerable intersections or sums, the operation $(\mathfrak{A})$, and an operation $(\mathfrak{C}\mathfrak{A})$ defined by the author. (3) Two sets in $C\bar{A}_2$ without common points may be separated by means of sets in $\bar{B}_2$. (4) If one suppresses the common part of two sets in $C\bar{A}_2$ (or in $\bar{A}_2$), the two remaining sets may be separated by means of sets in $\bar{A}_2$. Theorems 3 and 4 for sets A_2, CA_2 and B_2 are theorems of P. Novikoff [Fund. Math. **25**, 459–466 (1935)].
J. V. Wehausen (Falls Church, Va.).

Sierpiński, Wacław. Sur l'inversion du théorème de Bolzano-Weierstrass généralisé. Fund. Math. **34**, 155–156 (1947).

Let M be any metric space. A sequence $\{A_n\}_{n=1}^\infty$ of subsets of M is said to be convergent if, for every $p\epsilon M$ and for every open sphere S with center p, $S\cap A_n=0$ for almost all n or $S\cap A_n\neq 0$ for almost all n. The author proves that, if $2^{\aleph_0}=\aleph_1$, then every nonseparable metric space contains a sequence of subsets which has no convergent subsequence.
E. Hewitt (Chicago, Ill.).

Dvoretzky, A., and Motzkin, Th. The asymptotic density of certain sets of real numbers. Duke Math. J. **14**, 315–321 (1947).

Let $S(a,b)$ denote the Lebesgue measure of the intersection of the set S of real numbers with the interval (a,b). A set S is said to have the right (left) asymptotic density S^+ (S^-) if $\lim_{t\to+\infty}(S(0,t)/t)=S^+$, $\lim_{t\to+\infty}(S(-t,0)/t)=S^-$. Consider any measurable set P of real numbers. A real number a is termed an α (α^*) head if, for any $t>0$, $P(a,t)>(\geqq)\ \alpha\bar{P}(a,t)$, where $\bar{P}$ is the complement of P. The most general result proved by the authors is the following. The set S of α (α^*) heads is measurable; its right asymptotic density is $p^+-\alpha(1-p^+)$ for $0<\alpha\leqq p^+/(1-p^+)$ and 0 otherwise; its left asymptotic density is $p^--\alpha(1-p^-)$ for $0<\alpha<\mu=\min[p^-/(1-p^-),\ p^+/(1-p^+)]$ and 0 for $\alpha>\mu$; for $\alpha=\mu$ it is either 0 or $p^--\alpha(1-p^-)$ depending on conditions given by the authors. From this general result it follows for the case that P is periodic with, say, period 1 that $S(0,1)=P(0,1)-\alpha\bar{P}(0,1)$ for $P(0,1)-\alpha\bar{P}(0,1)\geqq 0$ and 0 otherwise. *H. B. Mann* (Columbus, Ohio).

Martinez Salas, J. Note on Riesz's lemma. Revista Mat. Hisp.-Amer. (4) **6**, 84–89 (1946). (Spanish)

The author extends a lemma, originally proved by F. Riesz for continuous functions [Acta Litt. Sci. Szeged **5**, 208–221 (1932); Saks, Theory of the Integral, Warszawa-Lwów, 1937, p. 142], to functions with only jump discontinuities, and to functions of n variables and of bounded variation according to the author's definition [Revista Mat. Hisp.-Amer. (4) **6**, 25–42 (1946); these Rev. **8**, 142]. The author points out that Riesz stated without proof the theorem for the functions with only jump discontinuities.
J. V. Wehausen (Falls Church, Va.).

Martinez Salas, J. The differentiation of functions of n real variables of bounded variation. Revista Mat. Hisp.-Amer. (4) **6**, 217–221, 249–253 (1946). (Spanish)

The author generalizes the definition of the Dini derivatives to a function of n real variables $f(x_1,\cdots,x_n)$, defined on an n-dimensional cube. The Dini derivatives defined correspond to the partial derivative $\partial^n f/\partial x_1\cdots\partial x_n$, but there are now 2^n upper and 2^n lower derivatives to be considered. In a previous paper [same Revista (4) **6**, 25–42 (1946); these Rev. **8**, 142] the author has given a definition of "bounded variation" for functions of n variables. He now shows that, if $f(x_1,\cdots,x_n)$ is of bounded variation in his sense, the Dini derivatives are measurable and finite except at most on a set of n-dimensional measure zero. His main theorem is that for any such function of bounded variation all the Dini derivatives coincide except on a set of n-dimensional measure zero. *J. V. Wehausen.*

de Possel, René. Les théories modernes de l'intégration. Revista Fac. Ci. Univ. Coimbra **15**, 29–39 (1946).

This is an expository article. The author sketches an abstract form of the Riemann-Stieltjes integral and outlines a technique of extension of the definition by extension of nonnegative functionals, a method similar to that explained in detail in McShane's "Integration" [Princeton University Press, 1944; these Rev. **6**, 43]. *M. M. Day* (Urbana, Ill.).

Nikodym, Otton-Martin. Remarques sur les intégrales de M. Jean-Louis Destouches considérées dans sa théorie des prévisions. C. R. Acad. Sci. Paris **225**, 479–481 (1947).

A function x from a Boolean σ-algebra T to a Banach space B is integrable on $a(\epsilon T)$ in the sense of Destouches if (1) $\sum x(b_i)$ converges for every disjoint sequence $\{b_i\}$ of elements of T and (2) the Moore-Smith limit of the sums $\sum x(b)$ for $b\epsilon D$ (where D is a countable partition of a and the ordering of D's is by refinement) exists in B. Using a slight extension of a remark of Orlicz concerning absolute convergence in Banach spaces, the author derives certain properties (such as countable additivity) of this integral.
P. R. Halmos (Princeton, N. J.).

Scheffé, Henry. A useful convergence theorem for probability distributions. Ann. Math. Statistics **18**, 434–438 (1947).

Soit $\{p_n(x)\}$ $(n=1,2,\cdots)$ une suite de densités de probabilité sur la droite; on a donc: $p_n(x)\geqq 0$, $\int_{-\infty}^{\infty}p_n(x)dx=1$; supposant que $\lim_{n\to\infty}p_n(x)=p(x)$ existe presque-partout, l'auteur démontre que si $\int_{-\infty}^{\infty}p(x)dx=1$, l'intégrale $\int_E p_n(x)dx$, où E est un ensemble mesurable-B fixe quelconque, tend vers $\int_E p(x)dx$, uniformément par rapport à E (il suffit évidemment que E soit mesurable-L); l'énoncé et la démonstration s'étendent immédiatement au cas de densités à un

nombre fini quelconque de dimensions. Le théorème est un corollaire immédiat du lemme de Fatou. *R. Fortet.*

Kestelman, H. Integral properties of non-measurable functions. J. London Math. Soc. 21 (1946), 283–290 (1947).

The interval $0 \leqq x \leqq 1$ is denoted by J. If $f(x) \geqq 0$, $x \epsilon S \subset J$, then $\Omega(f, S) = E[x \epsilon S, 0 \leqq y \leqq f(x)]$. The lower Lebesgue integral of f over S is the interior measure of $\Omega(f, S)$, denoted by $\underline{L}(f, S)$. The function $\varphi(x)$ is a Perron minor function of $f(x)$ on $[a, b]$ if (i) φ is continuous and $\varphi(a) = 0$; (ii) $\bar{D}\varphi \leqq f$ almost everywhere; (iii) $\bar{D}\varphi < \infty$. Then the lower Perron integral of f is $\sup \varphi(b) = \underline{P}(f, J)$. If $f \geqq 0$ then $\underline{L}(f, J) = \sup L(g, J)$, g measurable and $0 \leqq g \leqq f$. There is a measurable function γ satisfying $\underline{L}(f, J) = L(\gamma, J)$, $0 \leqq \gamma(x) \leqq f$, $x \epsilon J$. If $\underline{L}(f, J) < \infty$, $\nu(x)$ measurable and $\nu(x) \leqq f(x)$, then $\nu \leqq \gamma$ almost everywhere in J. If $f(x) \geqq 0$, $x \epsilon J$, then $\underline{P}(f, J) = \sup \varphi(1)$, φ absolutely continuous minor functions to f. Also $\underline{P}(f, J) = \underline{L}(f, J)$. If $F(x) = \underline{L}(f, 0, x)$ then $F(x)$ is absolutely continuous and $F'(x) = \gamma(x) \leqq f(x)$ almost everywhere. Let $L_*(x)$ be the lower bound of numbers a, where a is such that the set where $f(\xi) < a$, $\xi \epsilon J$, $\xi \neq x$ has metric density unity at x. Then $L_*(x) = \gamma(x)$ almost everywhere and $\underline{L}(f, J) = L(L_*, J) = L(\gamma, J)$. The methods and results of this paper are closely related to those of H. Blumberg [Acta Math. 65, 263–282 (1935)].

R. L. Jeffery (Kingston, Ont.).

Theory of Functions of Complex Variables

Kvesselava, D. A. Solution d'un problème limite de T. Carleman. C. R. (Doklady) Acad. Sci. URSS (N.S.) 55, 679–682 (1947).

Let L be a simple closed curve in the complex z-plane, with the angle of inclination of the tangent belonging to H (Hölder class); S^+ is the interior, S^- is the exterior, of L ($z = 0$ in S^+); let $\alpha'(t) \epsilon H$ (on L) and let $\alpha(t)$ realize a one-to-one representation of L on itself, so that t and $\alpha(t)$ describe L in opposite directions; $\varphi(z)$, analytic in S^+ (except at a finite number of poles) and admitting a continuous extension beyond L, is termed meromorphic in S^+. The author considers the problem of finding a $\varphi(z)$, meromorphic in S^+, so that (1) $\varphi^+(\alpha(t)) = G(t)\varphi^+(t) + g(t)$ (on L), where G, g are H on L and $G \neq 0$ on L. The homogeneous problem (1), with (2) $\alpha(\alpha(t)) = t$, has been studied by T. Carleman [Verh. Int. Math. Kongr., Zürich, 1932, v. 1, pp. 138–151]. Assuming (2) and using a different method, the author gives an explicit solution of the nonhomogeneous problem (1).

W. J. Trjitzinsky (Urbana, Ill.).

Satô, Tokui. Sur la conservation des angles dans la représentation conforme. Mem. Fac. Sci. Kyūsyū Imp. Univ. A. 3, 75–80 (1944).

Enoncé à peine différent d'un théorème de Wolff [Nederl. Akad. Wetensch., Proc. 38, 46–50 (1934)]. Le problème à été complétement résolu par Ostrowski [Prace Mat. Fiz. 44, 371–471 (1937)]. *J. Ferrand* (Caen).

af Hällström, Gunnar. Zur konformen Abbildung von Einschnittgebieten. Acta Acad. Aboensis 15, no. 1, 13 pp. (1944).

The author is concerned with the behavior of the Green's function of a region of the extended plane whose boundary consists of the closure of a denumerable number of disjoint rectilinear segments lying in $(-1, 1]$ and clustering solely at -1. The author associates with this Green's function in a natural way a region consisting of the interior of the unit circle slit along a denumerable number of radii and in this manner obtains results concerning the behavior of the Green's function in the neighborhood of the point -1. Use is made of previous results of the author [Acta Soc. Sci. Fennicae. Nova Ser. A. 3, no. 5 (1944); these Rev. 7, 447] and of E. Kaila [Ann. Acad. Sci. Fennicae (A) 55, no. 9 (1940); these Rev. 3, 80]. *M. Heins.*

Hayman, W. K. Some remarks on Schottky's theorem. Proc. Cambridge Philos. Soc. 43, 442–454 (1947).

The paper is devoted to numerical estimates in Schottky's theorem. It is based on a rather detailed discussion of the elliptic modular function by geometric methods. The author shows, in particular, that an analytic function $f(z) \neq 0, 1$ in $|z| < 1$ satisfies $|f(z)| \leqq \{e^\pi \max(1, |f(0)|)\}^{(1+\rho)/(1-\rho)}$ for $|z| = \rho$. The constant e^π is the best possible. Another formula is also derived which gives better results for small ρ.

L. Ahlfors (Cambridge, Mass.).

Erdös, P., and Fried, H. On the connection between gaps in power series and the roots of their partial sums. Trans. Amer. Math. Soc. 62, 53–61 (1947).

Let $f(x) = 1 + a_1 x + \cdots + a_n x^n$ have the radius of convergence 1. This series has Ostrowski gaps ρ if to this number ρ there corresponds a pair of infinite sequences m_k and n_k with $m_k < n_k$, $\lim n_k/m_k > 1$, such that $|a_n| < \rho^n$ $(m_k \leqq n \leqq n_k)$. It has infinite Ostrowski gaps ρ if to every $\rho' > \rho$ there corresponds a pair of infinite sequences m_k and n_k depending on ρ' with $m_k < n_k$ and $\lim n_k/m_k = \infty$, and such that $|a_n| < \rho' n$ for $m_k \leqq n \leqq n_k$. Let $A(n, r)$ denote the number of roots of $f(x) = 1 + a_1 x + \cdots + a_n x^n$. The authors give a new proof of Bourion's theorem [L'ultraconvergence dans les séries de Taylor, Actual. Sci. Indus., no. 472, Hermann, Paris, 1937] by which a necessary and sufficient condition that a power series has Ostrowski gaps is that there exists an $r > 1$, such that $\liminf_{n \to \infty} A(n, r)/n < 1$. They also prove that a necessary and sufficient condition that a power series has infinite Ostrowski gaps ρ is that $\liminf A(n, r)/n = 0$.

S. Mandelbrojt (Paris).

Valiron, Georges. Les fonctions entières d'ordre nul. Revista Unión Mat. Argentina 12, 168–180 (1947).

Valiron, Georges. Le théorème de Picard. Revista Unión Mat. Argentina 12, 181–193 (1947).

Lectures at the University of Buenos Aires.

Valiron, Georges. Directions de Julia et directions de Picard des fonctions entières. Revista Unión Mat. Argentina 12, 49–54 (1946).

This paper is concerned with certain refinements by Julia and Ostrowski of Picard's theorem on the values of an integral function. *O. Helmer* (Santa Monica, Calif.).

Ibraguimoff, I. Sur la convergence de la série interpôlatoire d'Abel-Gontcharoff. Rec. Math. [Mat. Sbornik] N.S. 21(63), 49–62 (1947). (Russian. French summary)

The series in question has the form

$$\sum_{k=0}^{\infty} \frac{1}{k!} F^{(k)}(\alpha_k) \int_{\alpha_0}^{z} dz_1 \int_{\alpha_1}^{z_1} \cdots \int_{\alpha_{k-1}}^{z_{k-1}} dz_k,$$

where $\alpha_k \geqq 0$. Let $s_n = \sum_{\nu=0}^{n-1} |\alpha_{\nu+1} - \alpha_\nu|$. The author uses an estimate for the remainder given by Gontcharoff [Ann. Sci.

École Norm. Sup. (3) **47**, 1–78 (1930)] to prove the following theorem. Let $n(r)$ denote the number of s_n in $(0, r)$; if, for some θ in $0<\theta<\frac{1}{2}$, $F(z)$ satisfies $\log M(r) \leqq C(\theta)n(\theta r)$ with $C(\theta)<\log \{(1-\theta)/\theta\}$, the series converges to $F(z)$ uniformly in every finite circle. The following corollaries are obtained. (I) If $\lim\sup n^{-1/\rho}s_n=\tau$ and $\sigma \leqq \omega^\rho(1+\omega)^{1-\rho}\rho^{-1}\tau^{-\rho}$, where ω is the positive root of $\omega^\rho e^{\omega+1}=1$, then the series represents all entire functions of growth less than ρ or of order ρ, type less than σ. (II) If $\lim s_n(\log n)^{1/\rho}=\tau$, $0 \leqq A<(2\tau)^{-\rho}$, the series represents all entire functions for which $\lim r^{-\rho}\log\log M(r)=A$. (I) and a less sharp form of (II) were given by Gontcharoff. The larger part of the paper is devoted to constructing an example to show that the main theorem fails if $\theta>\frac{1}{2}$. *R. P. Boas, Jr.*

MacLane, Gerald R. Concerning the uniformization of certain Riemann surfaces allied to the inverse-cosine and inverse-gamma surfaces. Trans. Amer. Math. Soc. **62**, 99–113 (1947).

The Riemann surfaces discussed are obtained by continuous deformations of those of $z=(\text{arc}\cos w)^2$, $z=\text{arc}\cos w$, $w=1/\Gamma(-z)$, such that the algebraic winding points are displaced along the real axis without crossing each other (on the surface), other singularities remaining fixed. Extending results of F. E. Ulrich [Duke Math. J. **5**, 567–589 (1939); these Rev. **1**, 8] these surfaces are all shown to be of parabolic type and the corresponding integral functions $w=f(z)$ are of order not exceeding 2. The greatest order can be reached only if the winding points are displaced "almost to coincidence" in a precisely defined manner. The nature of the canonical products and set of zeros of the functions $f'(z)$ are discussed. The converses of the results obtained are substantially true. The "fundamental regions" and curves of reality of the functions are also considered. The basis of the method used is an approximation by polynomials whose roots are located with sufficient accuracy by manipulation of inequalities. *A. J. Macintyre.*

Temliakoff, A. Prolongement analytique d'une fonction de deux variables. Rec. Math. [Mat. Sbornik] N.S. **19(61)**, 73–84 (1946). (Russian. French summary)

The author applies the method of integral operators to the study of functions of two complex variables. He shows that every function $\varphi(z_1, z_2)=\sum_{n,m}a_{nm}z_1^nz_2^m$ of two complex variables can be represented in the form

$$\varphi(z_1, z_2)=(2\pi)^{-1}\int_0^{2\pi} f(u, e^{-it})dt,$$

$$f=\int_0^1 \{d[u\varphi(uT), u(1-T)e^{-it}]/du\}dT.$$

Using the above representation he shows that from classical theorems on functions of one complex variable it follows that the series $\sum_{n,m}a_{nm}z_1^nz_2^m$ converges in the circular domain $C_R=[|x|+|y|<R]$, where

$$R^{-1}=\lim_{n\to\infty}\sup\left\{\sum_{k=0}^n (n!)^{-1}k!(n-k)!|a_{k,n-k}|\right\}^{1/n},$$

and that on the boundary of the largest domain C_R in which the above series converges there lies at least one singular point. In certain cases the author obtains relations between the behavior of the coefficients a_{nm} and the nature of the singularities on the boundary of C_R. [The reviewer wishes to add that the method could be used for the study of the questions of analytic continuation of the function φ outside of C_R. See, e.g., Bergman, Trans. Amer. Math. Soc. **53**, 130–155 (1943), p. 141; these Rev. **4**, 159.]

S. Bergman (Cambridge, Mass.).

Rocco Boselli, Anna. Le serie di potenze nelle sedici algebre complesse del 4. ordine dotate di modulo. Atti Accad. Sci. Napoli (3) **2**, no. 6, 39 pp. (1946).

Consider the hypercomplex variable $x=x_1u_1+\cdots+x_4u_4$, where $u_1, \cdots, u_4$ are basis elements of an associative algebra with a unit element over the complex field. The power series discussed by the author are of the types $\sum x^n\alpha_n$, $\sum \alpha_nx^n$, where the coefficients belong to the algebra. She investigates these series for each one of sixteen types of algebra, and in every case obtains conditions on the coefficients α_n which ensure that these power series define, within their region of convergence, functions which are analytic in the sense of Ringleb [Rend. Circ. Mat. Palermo **57**, 311–340 (1933)] and totally differentiable on the right or left in the sense of Spampinato [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (6) **21**, 621–625, 683–687 (1935)]. The proofs involve considerable computation which would be prohibitive for algebras of higher order.

S. A. Jennings (Vancouver, B. C.).

Theory of Series

Leja, F. Sur les suites monotones en moyenne. Ann. Soc. Polon. Math. **19** (1946), 133–139 (1947).

A sequence $\{a_n\}$ is called decreasing (increasing) in mean in the arithmetic sense if $a_{n+2}\leqq(\geqq)A(a_n, a_{n+1})$ $(n=1, 2, \cdots)$, where $A(x, y)$ is the arithmetic mean of x, y; and there are similar definitions (for positive sequences) relating to the geometric and harmonic means. It is proved that, if a sequence is monotonic in mean, it has a finite or infinite limit. The definition and theorem are extended in various ways. The limitations of the idea are indicated by the observation that, if $a_{n+1}\leqq A(a_1, a_2, \cdots, a_n)$ $(n=1, 2, \cdots)$, the sequence need not have any limit. *A. E. Ingham.*

Hamilton, Hugh J. Mertens' theorem and sequence transformations. Bull. Amer. Math. Soc. **53**, 784–786 (1947).

The paper treats pairs of convergent series $\sum a_i=A$, $\sum b_i=B$ and their Cauchy products. J. D. Hill has proved [Tôhoku Math. J. **45**, 332–337 (1939)] that Mertens' theorem [J. Reine Angew. Math. **79**, 182–184 (1874)] can be strengthened to read: a necessary and sufficient condition that the Cauchy product of a convergent series $\sum a_i$ and every convergent series $\sum b_i$ converges to AB is that $\sum|a_i|$ converges. The author uses Toeplitz transformations to obtain a simple and concise proof of Hill's version of the theorem. He also applies the method to doubly-infinite series to supplement recent results by I. M. Sheffer [Bull. Amer. Math. Soc. **52**, 1036–1041 (1946); these Rev. **8**, 260]. The combined results are as follows. A necessary and sufficient condition that the Cauchy product of the convergent series $\sum a_{ij}$ and every convergent (absolutely convergent) (boundedly convergent) series $\sum b_{ij}$ converges to AB is that the series $\sum a_{ij}$ terminates (converges boundedly) (converges absolutely). Convergence is in the sense of Pringsheim, i.e., by rectangular sums $A_{mn}=\sum a_{ij}$, $i=0, \cdots, m$; $j=0, \cdots, n$, as both m and n become large. The convergent series $\sum a_{ij}$ converges boundedly provided that the set of all partial sums A_{mn} is bounded.

G. Piranian (Ann Arbor, Mich.).

Rajagopal, C. T. A note on the oscillation of Riesz means of any order. J. London Math. Soc. 21 (1946), 275–282 (1947).

If s_n is a given sequence, the author considers the limits of oscillation $\underline{\sigma}_k$ of its Riesz means $\sigma_k(x)$ defined by $s(x)=s(\lambda_n)=s_n$ when $\lambda_n \leqq x<\lambda_{n+1}$, $0<\lambda_1<\cdots<\lambda_n\rightarrow\infty$; $\sigma_k(x)=x^{-k}s_k(x)=kx^{-k}\int_0^x(x-t)^{k-1}s(t)dt$, $k>0$; $\sigma_0(x)=s_0(x)=s(x)$; $\underline{\sigma}_k=\underline{\lim}_{x\rightarrow\infty}\sigma_k(x)$, $k\geqq 0$. He proves that, if

$$(1)\qquad \lambda_n(s_n-s_{n-1})\geqq -H_1(\lambda_n-\lambda_{n-1}),\qquad H_1>0,\ k\geqq 1,$$

then

$$\exp\left(\frac{\bar{\sigma}_k-\bar{\sigma}_{k+1}}{H_1}\right)-1-\frac{\bar{\sigma}_k-\underline{\sigma}_{k+1}}{H_1}\leqq 0,$$

$$\exp\left(\frac{\underline{\sigma}_k-\bar{\sigma}_{k+1}}{H_1}\right)-1-\frac{\underline{\sigma}_k-\underline{\sigma}_{k+1}}{H_1}\leqq 0.$$

Under the further condition

$$(2)\qquad \lambda_n(s_n-s_{n-1})\leqq H_2(\lambda_n-\lambda_{n-1}),\qquad H_2>0,$$

it is proved that

$$H_1\left\{\exp\left(\frac{\bar{\sigma}_k-\bar{\sigma}_{k+1}}{H_1}\right)-1\right\}+H_2\left\{\exp\left(\frac{\underline{\sigma}_{k+1}-\bar{\sigma}_k}{H_2}\right)-1\right\}\leqq 0,$$

$$H_1\left\{\exp\left(\frac{\underline{\sigma}_k-\bar{\sigma}_{k+1}}{H_1}\right)-1\right\}+H_2\left\{\exp\left(\frac{\underline{\sigma}_{k+1}-\underline{\sigma}_k}{H_2}\right)-1\right\}\leqq 0.$$

Moreover, these inequalities hold for $0\leqq k<1$ provided that either $\lim(\lambda_{n+1}/\lambda_n)=1$ or $\lim(s_n-s_{n-1})=0$ as $n\rightarrow\infty$.

H. R. Pitt (Belfast).

Hadwiger, H. Ueber eine Konstante Tauberscher Art. Revista Mat. Hisp.-Amer. (4) 7, 65–69 (1947).

The following theorem is given. There is a best (that is, least) constant K such that, for each series $a_1+a_2+\cdots$ with $n|a_n|$ bounded and with partial sums $s_1, s_2, \cdots$, to each $n=1, 2, \cdots$ corresponds a point t_n such that

$$(1)\qquad \left|\sum_{k=1}^{\infty}a_kt_n^k-s_n\right|\leqq K \operatorname*{l.u.b.}_{n=1,2,\cdots} n|a_n|$$

[$<$ is printed for $\leqq$ in this formula]. The best constant K is

$$(2)\qquad K=\gamma+\log\log 2+2\int_{\log 2}^{\infty}x^{-1}e^{-x}dx=.96804\cdots,$$

where γ is Euler's constant. Moreover, for this K,

$$(3)\qquad \limsup_{n\rightarrow\infty}\left|\sum_{k=1}^{\infty}a_kt_n^k-s_n\right|\leqq K\limsup_{n\rightarrow\infty}n|a_n|$$

when $t_n=2^{-1/n}$.

These results are closely related to a theorem of the reviewer [Duke Math. J. 12, 27–36 (1945); these Rev. 7, 12] about the set L of limit points of a sequence s_n and the set L_A of limit points of its Abel transform $\sum a_nt^n$. The same K is the best constant such that to each z' in L corresponds a z'' in L_A such that

$$(4)\qquad |z'-z''|\leqq K\limsup n|a_n|.$$

The reviewer gave the value $K=.9680448\cdots$. It seems that there has been no determination of the best constant K' such that to each z'' in L_A corresponds a z' in L such that (4) holds when $K=K'$; for incomplete results, see Hadwiger [Comment. Math. Helv. 16, 209–214 (1944); these Rev. 5, 236] and the paper of the reviewer cited above. See also the following review.

R. P. Agnew.

Hartman, Philip. Tauber's theorem and absolute constants. Amer. J. Math. 69, 599–606 (1947).

Two "best" absolute constants τ and τ^* are determined such that the inequalities

$$(1)\qquad \limsup_{t\rightarrow 1-}\left|\sum_{n=1}^{\infty}a_nt^n-\sum_{n=1}^{-1/\log t}a_n\right|\leqq\tau\limsup_{n\rightarrow\infty}n^{-1}\left|\sum_{k=1}^{n}ka_k\right|$$

and

$$(2)\qquad \limsup_{t\rightarrow 1-}\left|\sum_{n=1}^{\infty}a_nt^n-\sum_{n=1}^{-1/\log t}a_n\right|\leqq\tau^*\limsup_{n\rightarrow\infty}|na_n|$$

hold whenever $a_1, a_2, a_3, \cdots$ is a real sequence such that $\sum a_nt^n$ converges when $|t|<1$. The best, that is, least, possible values of τ and τ^* are shown to be $\tau=1.75174\cdots$ and $\tau^*=1.01598\cdots$.

The author cites a constant ρ defined by Hadwiger [Comment. Math. Helv. 16, 209–214 (1944); these Rev. 5, 236] and says it is clear from Hadwiger's proof of the existence of ρ that $\rho=\tau^*$. This statement contradicts results of Agnew [Duke Math. J. 12, 27–36 (1945); these Rev. 7, 12] and Hadwiger [see the preceding review] which imply that a smaller constant can be placed in the right member of (2) when the upper limit in $\sum a_n$ in the left member is replaced by the more favorable $n(t)=[\log 2/\log t^{-1}]$. In fact, these authors showed, respectively, that

$$\limsup_{t\rightarrow 1-}\left|\sum_{k=1}^{\infty}a_kt^k-\sum_{k=1}^{n(t)}a_k\right|\leqq\rho\limsup_{n\rightarrow\infty}n|a_n|,$$

$$\limsup_{n\rightarrow\infty}\left|\sum_{k=1}^{\infty}a_k(2^{-1/n})^k-\sum_{k=1}^{n}a_k\right|\leqq\rho\limsup_{n\rightarrow\infty}n|a_n|$$

when ρ has the value $.96804\cdots$. The constants τ, τ^*, and ρ are expressed in terms of Euler's constant and transcendental definite integrals. [Cf. the two following reviews.]

R. P. Agnew (Ithaca, N. Y.).

Wintner, Aurel. On Tauber's theorem. Comment. Math. Helv. 20, 216–222 (1947).

There exist constants τ and τ^* such that the inequalities (1) and (2) of the preceding review hold for each series $a_1+a_2+\cdots$ for which $\sum a_nt^n$ converges when $|t|<1$. Lower and upper estimates of the best (that is, least) possible values of τ and τ^* are obtained. Knowing this work of Wintner before it was published, Hartman [see the preceding review] determined the best values of τ and τ^*. The constants τ and τ^* are also treated by Hadwiger [see the following review].

R. P. Agnew (Ithaca, N. Y.).

Hadwiger, H. Die Retardierungserscheinung bei Potenzreihen und Ermittlung zweier Konstanten Tauberscher Art. Comment. Math. Helv. 20, 319–332 (1947).

Wintner [see the preceding review] showed the existence of least constants τ^* and τ. The author obtains the values of τ and τ^*, as was also done by Hartman [see the second preceding review]. Specific series $\sum a_n$ are given for which equality holds in (1) and (2) [see the second preceding review] when τ^* and τ have the least values. The same constants τ^* and τ are the least constants such that

$$(3)\qquad \limsup_{n\rightarrow\infty}\left|\sum_{k=1}^{\infty}a_k(1-n^{-1})^k-\sum_{k=1}^{n}a_k\right|\leqq\tau^*\limsup_{n\rightarrow\infty}n|a_n|,$$

$$(4)\qquad \limsup_{n\rightarrow\infty}\left|\sum_{k=1}^{\infty}a_k(1-n^{-1})^k-\sum_{k=1}^{n}a_k\right|\leqq\tau\limsup_{n\rightarrow\infty}n^{-1}\left|\sum_{k=1}^{n}ka_k\right|,$$

whenever $\sum a_kt^k$ converges when $|t|<1$. For each n, best

constants τ_n^* and τ_n are determined such that

$$\left|\sum_{k=1}^{\infty} a_k(1-n^{-1})^k - \sum_{k=1}^{n} a_k\right| \leqq \tau_n^* \ \underset{n=1,2,\cdots}{\text{l.u.b.}}\ n|a_n|,$$

$$\left|\sum_{k=1}^{\infty} a_k(1-n^{-1})^k - \sum_{k=1}^{n} a_k\right| \leqq \tau_n \ \text{l.u.b.}\ (n+1)^{-1}\left|\sum_{k=1}^{n} ka_k\right|,$$

whenever $\sum a_k t^k$ converges when $|t|<1$. It is shown that the best constants in (3) and (4) are limits of these constants, and these results are used to prove the facts relating to (1) and (2). *R. P. Agnew* (Ithaca, N. Y.).

Northcott, D. G. Abstract Tauberian theorems with applications to power series and Hilbert series. Duke Math. J. 14, 483–502 (1947).

Let $\{a_n\}$ belong to a fixed (B)-space and $\|a_n\|=O(n^{\gamma-1})$, $\gamma\geqq 0$. Let $(1-x)^\gamma\sum_0^\infty a_n x^n\to\zeta$ when $x\to 1-0$. Then $\Gamma(\gamma+1)n^{-\gamma}\sum_0^n a_k\to\zeta$ when $n\to\infty$. The author gives a direct proof along the lines of Karamata for this abstract form of a classical Tauberian theorem. [It also follows from the reviewer's extension of Wiener's Tauberian theorem, as observed by the author.] Among the applications, the following should be noted. (I) If a_n is complex, if $a_n=O(n^{1/p-1})$, if $s(x)=\sum_0^\infty a_n x^n$ belongs to $L_p(0,1)$, then $\|s(x)-s_n(x)\|_p\to 0$ when $n\to\infty$, where $s_n(x)$ is the nth partial sum of the power series. (II) If a_n is real, if $a_n=O(n^{\frac{1}{2}\gamma-1})$, $\gamma>0$, and

$$J_\gamma=(1/\Gamma(\gamma))\int_0^1\{\log(1/x)\}^{\gamma-1}s^2(x)dx,$$

then $A_{mn}=\sum_0^m\sum_0^n a_\mu a_\nu/(\mu+\nu+1)^\gamma\to J_\gamma$ when $m,n\to\infty$ independently. Here $J_\gamma\leqq\infty$. It is remarkable that the double series does not show any other behavior than unconditional convergence or divergence. The case $\gamma=1$ gives Hilbert's series. Replacing the factor $\{\log(1/x)\}^{\gamma-1}$ by $(1-x)^{\gamma-1}$ in the definition of J_γ in (II), the author proves a similar convergence theorem for a double series obtained by replacing the factor $(\mu+\nu+1)^{-\gamma}$ in A_{mn} by a gamma-quotient. *E. Hille* (New Haven, Conn.).

Korevaar, J. An elementary proof of a Tauberian theorem for Lambert series. Simon Stevin 25, 83–114 (1947). (Dutch)

The author observes that the closure of certain systems of functions in $C[0,1]$ leads to Tauberian theorems. He proves the following Tauberian theorem for Lambert series due to S. Bochner [S.-B. Preuss. Akad. Wiss. Phys.-Math. Kl. 1933, 126–144] using an argument somewhat similar to that of J. Karamata [Math. Z. 32, 319–320 (1930)] for the original O-theorem of Littlewood. (I) If $h(x)=\sum_1^\infty a_n nx^n(1-x^n)^{-1}$ converges for $0\leqq x<1$, if for a certain $\alpha>0$ we have $(1-x)^{\alpha+1}h(x)\to B$ when $x\to 1-0$, and if $a_n>-Kn^{\alpha-1}$, then $n^{-\alpha}\sum_1^n a_k\to A$, where $A\alpha\Gamma(\alpha+1)\zeta(\alpha+1)=B$. The proof is based upon: (II) the system 1, $(1-x)^\beta x^{k-1}(1-x^k)^{-1}$, $k=1,2,3,\cdots$ is closed in $C[0,1]$ when $\beta>1$. The following Tauberian theorems are also proved. (III) If the system $\{f(x^k)x^{-1}\}$ is closed in $C[0,1]$, if $a_n>-Kn^{\alpha-1}$, $\alpha>0$, and if

$$(1)\qquad (1-x)^\alpha\sum_1^\infty a_n f(x^n)\to A\alpha\int_0^1(\log t^{-1})^{\alpha-1}f(t)t^{-1}dt,$$

the series being convergent for $0\leqq x<1$, then for each bounded Riemann integrable function $g(x)x^{-1}$ the same limit relation holds with $f(\cdot)$ replaced by $g(\cdot)$. (IV) If, under the same assumptions on $f(x)$ and a_n as in (III), the limit in (1) exists and equals B, then $n^{-\alpha}\sum_1^n a_k\to A$, where A is so chosen that the right member in (1) equals B. The author also shows that (I) can be derived from the corresponding Tauberian theorem for power series, due to Hardy and Littlewood, using the special Lambert series for x^n obtainable from the inversion formula of Möbius. Several other applications of the latter are given. *E. Hille.*

Chelidze, V. G. A theorem on double power series. C. R. (Doklady) Acad. Sci. URSS (N.S.) 53, 691–694 (1946).

Let $\sum a_{jk}$ be a double series with partial sums S_{mn} convergent to S. Let $S_{mn}/m\to 0$ for each n, let $S_{mn}/n\to 0$ for each m and let $S_{mn}/mn\to 0$ as $m,n\to\infty$. Let $\lambda>1$. Then the double series in $f(x,y)=\sum_{j,k=0}^\infty a_{jk}x^jy^k$ converges absolutely when $|x|<1$, $|y|<1$, and defines a function $f(x,y)$ such that $f(x,y)\to S$ as $x\to 1$, $y\to 1$ subject to the restrictions $0<x<1$, $0<y<1$ and $\lambda^{-1}<(1-x)/(1-y)<\lambda$.

The hypothesis that $S_{mn}/n\to 0$ for each m cannot be replaced by the hypothesis that $|S_{mn}|\leqq n+1$; this is shown by consideration of the series $\sum a_{jk}$ leading to the function $f(x,y)=(1-x)/(1-y)$. The restriction $\lambda^{-1}<(1-x)/(1-y)<\lambda$ cannot be removed from the conclusion of the theorem; this is shown by consideration of the series $\sum a_{jk}$ for which $a_{0,k}=(k+1)^{\frac{1}{2}}-k^{\frac{1}{2}}$, $k=0,1,2,\cdots$; $a_{1,k}=-a_{0,k}$, and $a_{jk}=0$ when $j>1$. *R. P. Agnew.*

Schweitzer, Miklós. Sur les produits infinis et le théorème d'Abel. Acta Univ. Szeged. Sect. Sci. Math. 11, 139–146 (1947).

Let $P(z)=\prod_1^\infty(1+a_nz^{k_n})$, where $\{k_n\}$ is a given never-decreasing sequence of positive integers. The theorem of Abel for such a product would be an assertion that the convergence of $P(1)=\prod_1^\infty(1+a_n)$ implies that $\lim_{z\to 1}P(z)$ exists and equals $P(1)$. By suitable examples G. H. Hardy [Proc. London Math. Soc. (2) 7, 40–48 (1909)] has shown that no such theorem holds if either $k_n\equiv 1$ or $k_n=n$. In this note the author proves that a necessary and sufficient condition for the validity of the Abelian theorem is that $k_1+\cdots+k_n\leqq Ck_n$ for each n. In the conclusion it is not necessary that $z\to 1$ radially; approach in a sector is also admissible. If the condition on the exponents is satisfied, then the converse Tauberian theorem also holds, provided $a_n\to 0$, that is, if $P(z)\to p\neq 0$ when $z\to 1$ and $a_n\to 0$ then $P(1)$ converges to p. *E. Hille* (New Haven, Conn.).

Fourier Series and Generalizations, Integral Transforms

Langer, Rudolph E. Fourier's Series. The Genesis and Evolution of a Theory. The first Herbert Ellsworth Slaught Memorial Paper. Amer. Math. Monthly 54, no. 7, part II. v+86 pp. (1947).

This is an elementary and well written exposition of some chapters from the theory of the representation of arbitrary functions by infinite series. The paper falls into two distinct parts. The first deals with the early history of the theory of trigonometric series up to and including Fourier's work. The problem of the vibrating string, as treated by John and David Bernoulli, Euler and d'Alembert, is discussed in some detail and the subsequent controversies concerning the nature of an arbitrary function are described. This is followed by an account of Fourier's method of treating the problem of the conduction of heat in a bar. The second part of the paper deals with the representation of an arbitrary function by more general orthogonal series, as they arise

in the theory of differential equations of the second order with boundary conditions. A sketch of the general theory of the characteristic values (Eigenwerte) is given, and a few examples are worked out in some detail.

W. W. Rogosinski (Newcastle-upon-Tyne).

Salem, R., and Zygmund, A. On a theorem of Banach. Proc. Nat. Acad. Sci. U. S. A. **33**, 293–295 (1947).

Consider a sequence of positive integers $\{n_k\}$ such that $n_{k+1}/n_k > q > 1$. The following theorems are known. (i) Given any sequence of real numbers $\alpha_1, \beta_1, \alpha_2, \beta_2, \cdots$ tending to 0, there exists a summable function

$$f(x) \sim \sum (a_n \cos nx + b_n \sin nx)$$

such that $a_{n_k} = \alpha_k$, $b_{n_k} = \beta_k$ $(k = 1, 2, \cdots)$. (ii) If, in addition, $\sum(\alpha_k^2 + \beta_k^2) < \infty$, then $f(x)$ may be chosen continuous. Theorem (i) was found independently by Sidon and Banach; (ii) is due to Banach. While there exist simple proofs of (i), namely that of Sidon [Math. Z. **34**, 477–480; **35**, 624 (1932); cf. Zygmund, Trigonometrical Series, Warszawa-Lwów, 1935, p. 220], based on the consideration of the products $\prod_1^\infty (1 + A_k(x))$, where $A_k(x) = \alpha_k \cos n_k x + \beta_k \sin n_k x$, the existing proofs of (ii) are long and complicated [cf. Zygmund's book, pp. 215 f.]. In this note it is shown that, by using the modified products $-i\prod_1^\infty (1 + iA_k(x))$, (ii) may be proved in a simple way, very analogous to that used by Sidon in proving (i).

B. de Sz. Nagy (Szeged).

Zygmund, A. On trigonometric integrals. Ann. of Math. (2) **48**, 393–440 (1947).

For some time many have believed that the convergence properties of trigonometric integrals were practically identical with the corresponding properties of trigonometric series. The results of this paper establish many conjectures in this field. We quote the following theorems.

Let $\phi(u)$ be of bounded variation in every finite range and such that $\alpha(u) = \sup_{0 \leq h \leq 1} |\phi(u+h) - \phi(u)|$ tends to zero as $u \to \infty$. Let (α, β) be any interval of length less than 2π. Then there is a trigonometric series $\sum_{-\infty}^{\infty} c_n e^{inx}$ with coefficients tending to zero and such that, as $\omega \to \infty$, the difference

$$\sum_{-\omega \leq n \leq \omega} c_n e^{inx} - \int_{-\omega}^{\omega} e^{inx} d\phi(u)$$

converges uniformly to zero in the interval $\alpha \leq x \leq \beta$. The author gives many extensions of this result. In the first place $\phi(u)$ need not be of bounded variation in every finite range if the integrals are interpreted in a generalized sense. Also the condition on $\alpha(u)$ can be relaxed but then summability may replace convergence. He also gives the corresponding results for conjugate series and integrals.

If E is a periodic set of period 2π and if every trigonometric series which converges to a function $f(x)$ outside E is the Fourier series of $f(x)$, if $\phi(u)$ satisfies the same conditions as before, and if the integral $\int_{-\infty}^{\infty} e^{ixu} d\phi(u)$ converges to $f(x)$ outside E, then

$$\phi(u_0 + \rho) - \phi(u_0) = \lim_{\omega \to \infty} (2\pi)^{-1} \int_{-\omega}^{\omega} f(x) e^{-iu_0 x} \frac{e^{-ix\rho} - 1}{-ix} dx$$

for some u_0 and almost every ρ; the limit on the right may be ordinary or generalized.

A. C. Offord.

Zygmund, A. A remark on characteristic functions. Ann. Math. Statistics **18**, 272–276 (1947).

Etant donnée la caractéristique $\varphi(v)$ d'une fonction de répartition $F(x)$, et à propos du problème de savoir si l'existence de la dérivée $\varphi'(0)$ entraine celle de (1) $\lim_{X \to +\infty} \int_{-X}^{X} x\,dF(x)$, l'auteur introduit la définition suivante: une fonction $\psi(v)$ définie au voisinage de la valeur v_0 est dite "lisse" en v_0 si

$$\lim_{h \to +0} h^{-1}\{\psi(v_0+h) + \psi(v_0-h) - 2\psi(v_0)\} = 0.$$

Il démontre que: si $\varphi(v)$ est lisse au point 0, une condition nécessaire et suffisante pour que $\varphi'(0)$ existe est que la limite (1) existe. Ce théorème résoud donc par l'affirmative le problème signalé plus haut; l'auteur établit également par la même méthode le théorème analogue pour les dérivées de $\varphi(v)$ et les moments de $F(x)$ d'ordre supérieur quelconque.

R. Fortet (Caen).

Misra, M. L. On the determination of the jump of a function by its Fourier coefficients. Quart. J. Math., Oxford Ser. **18**, 147–156 (1947).

Let $f(t) \sim \sum_1^\infty b_n \sin nt$, $\psi(t) = f(t) - d$, d a constant; let $L(\omega) = (\log \omega)^{-1} \sum_{n \leq \omega} b_n \log (\omega/n)$. The main result of this paper can be stated as follows: if $\int_t^\pi |\psi(u)| u^{-1} du = o(\log 1/t)$, as $t \to 0$, then $L(2\omega) - L(\omega) \to d\pi^{-1} \log 2$, and

$$\sum_{n \leq \omega} b_n \sim 2\pi^{-1} d \log \omega,$$

as $\omega \to \infty$. References to previous results are given.

O. Szász (Cincinnati, Ohio).

Boas, R. P., Jr. Inequalities for the coefficients of trigonometric polynomials. Nederl. Akad. Wetensch., Proc. **50**, 492–495 = Indagationes Math. **9**, 298–301 (1947).

Let $F(t) = \sum_{-n}^{n} a_k e^{ikt}$ be a trigonometric polynomial of order n and suppose for simplicity that $F(t)$ is real. Write $M = \max |F(t)|$ and $I = (2\pi)^{-1} \int_0^{2\pi} |F(t)| dt$. The author proves a number of inequalities about the coefficients a_k from which we quote the following: $|a_k| \leq M \cos \pi/(p+2)$, where $p > 0$ and p is the largest odd integer with $pk \leq n$; $|a_0| + |a_k| \sec \pi/([n/k]+2) \leq M$ for $k > 0$; $|a_0| + \frac{4}{3}|a_k| \leq (1 + \frac{1}{3}\sqrt{2})I$ for $k > n/2$ and $|a_0| + 2|a_k| \leq (1 + \sqrt{2})I$ for $k > n/2$. These inequalities are improvements on those of van der Corput and Visser [same Proc. **49**, 383–392 (1946); these Rev. **8**, 148]. The author combines the ideas of van der Corput and Visser with those of Boas and Kac [Duke Math. J. **12**, 189–206 (1945); these Rev. **6**, 265].

A. C. Offord (Newcastle-upon-Tyne).

Maruyama, Gisirô, and Kawata, Tatsuo. Interpolation. I. Mem. Fac. Sci. Kyûsyû Imp. Univ. A. **2**, 205–215 (1942).

It is known that the interpolation polynomials

$$U_n(f, x) = \frac{1}{2n+1} \sum_{i=0}^{2n} f(x_i) \frac{\sin (n+\frac{1}{2})(x - x_i)}{\sin \frac{1}{2}(x - x_i)}, \quad x_i = \frac{2\pi i}{2n+1},$$

which agree with $f(x)$ at the $2n+1$ points x_i, do not usually converge to $f(x)$ in $<0, 2\pi>$, even when f is continuous. It is shown that the expression

$$\tfrac{1}{2}\{U_n(f, x + p\pi/2n) + U_n(f, x - p\pi/2n)\}, \qquad p \text{ odd},$$

tends to $f(x)$, uniformly in $<0, 2\pi>$, when f is continuous there. [Actually, the proof shows that convergence to $f(x)$ holds wherever f is continuous, provided that f is bounded in $<0, 2\pi>$.] A similar result has been proved by G. Grünwald [Bull. Amer. Math. Soc. **47**, 271–275 (1941); these Rev. **2**, 283]. If f is of bounded variation in $<0, 2\pi>$, it is known that $U_n(f, x) \to f(x)$ wherever f is continuous. The authors discuss the limiting values of U_n at a point of discontinuity: the behavior is different according as x is a rational or irrational multiple of π. If f is of bounded vari-

ation, and $f(x)=\frac{1}{2}\{f(x+0)+f(x-0)\}$ throughout $\langle 0, 2\pi\rangle$, then the arithmetic means of $U_n(f)$ tend to f everywhere.
W. W. Rogosinski (Newcastle-upon-Tyne).

Maruyama, Gisirô, and Kawata, Tatsuo. Interpolation. II. Mem. Fac. Sci. Kyûsyû Imp. Univ. A. **3**, 57–65 (1943).

[Cf. the preceding review.] Let $f(x)$ be L-integrable in $\langle 0, 2\pi\rangle$, and let

$$F_n(x)=(2\delta_n)^{-1}\int_{x-\delta_n}^{x+\delta_n} f(u)du, \quad \delta_n=\pi/(2n+1).$$

It has been proved by A. C. Offord [Duke Math. J. **6**, 505–510 (1940); these Rev. **1**, 328] that the interpolation polynomials $U_n(F_n, x)$ [with notation as in the preceding review] tend to $f(x)$ in the Lebesgue set, that is, when

$$(*)\qquad \int_0^t |\varphi(u)|du=o(t),$$
$$\varphi(u)=\tfrac{1}{2}\{f(x+u)+f(x-u)\}-f(x).$$

The authors prove the same but under unnecessary additional assumptions. They also discuss the nth partial sums $t_n(x)$ of the Fourier series of $F_n(x)$ and show that, under wide conditions, these behave very much like the partial sums $s_n(x)$ of the Fourier series of $f(x)$.
W. W. Rogosinski (Newcastle-upon-Tyne).

Rappoport, S. Sur un procédé d'approximation des fonctions par polynômes trigonométriques. C. R. (Doklady) Acad. Sci. URSS (N.S.) **56**, 9–10 (1947).

Let $f(x)$ be defined for all x in the closed interval $(0, 2\pi)$ and set

$$V_n(f;x)=\frac{(2n)!!}{(2n+1)!!}\sum_{k=0}^{2n} f(x_k^{(n)})\cos^{2n}\tfrac{1}{2}(x_k^{(n)}-x),$$

where $x_k^{(n)}=2\pi k/(2n+1)$. Several results concerning the approximation of $f(x)$ by $V_n(f;x)$ are stated without proof. A typical result is the following: if $f(x)$ is bounded and $f''(x_0)$ exists (and is finite), then $\lim n\{V_n(f;x_0)-f(x_0)\}=f''(x_0)$ as $n\to\infty$. Another result is the following: if $f(x)\epsilon L$ and

$$\bar{V}_n(f;x)=\frac{(2n)!!}{(2n+1)!!}\sum_{k=0}^{2n}\left\{\frac{2n+1}{2\pi}\int_{x_k^{(n)}}^{x_{k+1}^{(n)}} f(t)dt\right\}\cos^{2n}\tfrac{1}{2}(x_k^{(n)}-x),$$

then $\bar{V}_n(f;x_0)\to f(x_0)$ at each x_0 such that

$$f(x_0)=\lim_{h\to 0} h^{-1}\int_{x_0}^{x_0+h} f(t)dt.$$

Finally, the definition of $V_n(f;x)$ is modified so as to conclude that $V_n(f;x)\to f(x)$ at each point of approximate continuity of $f(x)$. However, the reviewer was unable to understand this modified definition. *M. Kac.*

Sokól-Sokolowski, K. On trigonometric series conjugate to Fourier series of two variables. Fund. Math. **34**, 166–182 (1947).

The author extends the notion of conjugate series and of the associated conjugate function from one to several variables. Adopting the rectangular method of summation, and taking the typical case of two variables, he proves the following principal results. Theorem 1. If $f(x, y)$ is of period 2π in x and in y, and belongs to the Lebesgue class L^p, $p>1$, then the function

$$(*)\qquad \tilde{f}(x,y)=\lim_{\epsilon,\eta\to 0}\pi^{-2}\int_{-\pi}^{(\epsilon)\pi}\int_{-\pi}^{(\eta)\pi}\frac{f(x+u,y+v)dudv}{2\tan\frac{1}{2}u\cdot 2\tan\frac{1}{2}v}$$

(the symbolism indicates that integration is extended over the domain $\epsilon\leqq|u|\leqq\pi$, $\eta\leqq|v|\leqq\pi$) exists for almost all points (x, y), and satisfies the inequality

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}|\tilde{f}|^p dxdy\leqq A_p\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}|f|^p dxdy,$$

where A_p denotes a constant depending only on p. Theorem 2. If $f(x, y)$ belongs to the class $L^{1,k}$, where $k\geqq 3$, then $\tilde{f}(x, y)$ as defined in (*) exists almost everywhere, and belongs to the class $L^{1,k-2}$ (we have $f(x)\epsilon L^{p,k}$ if $|f(x)|^p\{\log^+|f(x)|\}^k$ is Lebesgue integrable). Theorem 1 is the two-dimensional analogue of a well-known result of M. Riesz [Zygmund, Trigonometrical Series, Warszawa-Lwów, 1935, p. 147].
K. Chandrasekharan (Princeton, N. J.).

Zahorski, Z. Sur les ensembles des points de divergence de certaines intégrales singulières. Ann. Soc. Polon. Math. **19** (1946), 66–105 (1947).

The paper contains, among others, proofs of results announced earlier [C. R. Acad. Sci. Paris **223**, 399–401, 415–417, 449–451, 465–467 (1946); these Rev. **8**, 264, 141, 148]. The main problem discussed is that of necessary and sufficient conditions for the sets of points where an integral

$$\int_a^b f(x+t)K(s,t)dt$$

depending on a variable x and parameter s does not tend to a limit as $s\to\infty$. The kernel $K(s, t)$ is supposed to satisfy various conditions. The following corollaries of general results obtained by the author are of interest. A necessary and sufficient condition that a set E of real numbers is the set of points of nondifferentiability of a function $f(x)\epsilon\mathrm{Lip}\,1$ is that E is a $G_{\delta\sigma}$ of measure 0. A similar result holds for the sets of points where the Fourier series of a bounded function is not summable (C, r), $r>0$, and the sets of points where the Poisson integral of a bounded function does not have a radial limit. *A. Zygmund* (Chicago, Ill.).

Souriau, Jean-Marie. Valeurs moyennes et transformation de Laplace. C. R. Acad. Sci. Paris **225**, 25–26 (1947).

The author defines mean values of fractional order of a given function $g(x)$ by $g_\alpha(x)=\alpha\int_0^1(1-t)^{\alpha-1}g(xt)dt$, $\Re(\alpha)>0$, and states consistency theorems for the case in which $g_\alpha(x)$ tends to a limit. He gives applications to the complex inversion formula of the Laplace transform $F(z)=z\int_0^\infty e^{-zt}f(t)dt$. Here the observation that the function

$$\psi(x)=\tfrac{1}{2}\{e^{zt}F(z)+e^{\bar{z}t}F(\bar{z})\},\qquad z=c+ix,$$

tends in the mean of order two to the saltus $f(t+0)-f(t-0)$ when $x\to\infty$, as well as the resulting applications to Dirichlet series, may be new. The remainder appears to be known: for the consistency theorems cf. Hardy and Riesz, The General Theory of Dirichlet's Series, Cambridge University Press, 1915, pp. 29–30; for the $(C, 1)$-summability of the inversion integral see D. V. Widder, The Laplace Transform, Princeton University Press, 1941, p. 77; these Rev. **3**, 232. *E. Hille* (New Haven, Conn.).

Widder, D. V. Inversion formulas for convolution transforms. Duke Math. J. **14**, 217–249 (1947).

The author gives detailed proofs of results announced earlier [Proc. Nat. Acad. Sci. U. S. A. **33**, 31–34 (1947); these Rev. **8**, 458]. Since the present paper has appeared better results have been obtained by the author [cf. the following review]. *H. Pollard* (Ithaca, N. Y.).

Widder, D. V. The inversion of a generalized Laplace transform. Proc. Nat. Acad. Sci. U. S. A. **33**, 295–297 (1947).

The author announces new results which supersede his previous work on convolution transforms [cf. the preceding review]. If $\phi(x)$ is continuous and absolutely integrable on $(-\infty, \infty)$, if $E(s)$ is defined by

$$E(s)=s^p\prod_{k=1}^{\infty}(1-s/a_k)e^{s/a_k}, \qquad 0<|a_1|\leqq|a_2|\leqq\cdots,$$

where $\sum_{k=1}^{\infty}1/a_k^2<\infty$, $G(x)$ by

$$G(x)=(2\pi i)^{-1}\int_{c-i\infty}^{c+i\infty}\frac{e^{sx}}{E(s)}ds, \qquad 0<c<|a_1|,$$

and $f(x)$ by $f(x)=\int_{-\infty}^{\infty}G(x-t)\phi(t)dt$, then

$$(*) \qquad \lim_{n\to\infty} D^p\prod_{k=1}^{n}(1-D/a_k)f(x+s_n)=\phi(x).$$

The method of proof is not given, but apparently no use is made of recent work of Schoenberg [Proc. Nat. Acad. Sci. U. S. A. 33, 11–17 (1947); these Rev. 8, 319] from which a proof is easily constructed. *H. Pollard* (Ithaca, N. Y.).

Polynomials, Polynomial Approximations

Ankeny, N. C. One more proof of the fundamental theorem of algebra. Amer. Math. Monthly **54**, 464 (1947).

If the real polynomial $P(z)$ of degree at least 2 has no roots, $\int_C P(z)^{-1}dz=0$, where C is a large semicircle with diameter on the real axis. Letting the radius become infinite, we have $\int_{-\infty}^{\infty}P(x)^{-1}dx=0$ and hence a contradiction since $P(x)$ cannot change sign. *R. P. Boas, Jr.*

Petterson, Erik L. Eine obere Grenze der Gradzahlen ganzer rationaler Funktionen als Folgerung funktionentheoretischer Beziehungen. Ark. Mat. Astr. Fys. **34B**, no. 3, 8 pp. (1947).

In this paper relations are developed between the degree n of a polynomial $h(z)=z^n+b_1z^{n-1}+\cdots+b_n$ and $M=\max|h(z)|$ on the curve G: $|g(z)|=c$, where $0<c$ and $g(z)=z^m+a_1z^{m-1}+\cdots+a_m$. From the lemma that, if $n\leqq m$, $\max|h(z)/g(z)|$ on or outside G equals M/c, it is concluded that $c^{n/m}\leqq M$. Furthermore, if $n<m$ and if t is any zero of $f(z)=g(z)+h(z)$, then $|g(t)|=|h(t)|\leqq M^{m/(m-n)}$. The latter result is applied to the case $g(z)=z^m$, thus showing that, if t is any zero of a polynomial $f(z)=z^m+A_1z^{m-1}+\cdots+A_m$ with $|A_1|=1$, then $|t|\leqq\max|f(z)-z^m|$, $|z|=1$. Finally, it is proved that, if $|g(z)|=R$ for at least one point on the circle $|z|=R$ and if $1\leqq|g(z)|$ for all z, $r\leqq|z|$, where $1<r<R$, then $m\leqq(\log R)/(\log(R/r))$. *M. Marden.*

Frank, Evelyn. On the real parts of the zeros of complex polynomials and applications to continued fraction expansions of analytic functions. Trans. Amer. Math. Soc. **62**, 272–283 (1947).

The author proves that, if $P(z)$ is an nth degree polynomial, and if $P^*(z)=\bar{P}(-z)$, then the $(n-1)$th degree polynomial $P_1(z)=[P^*(t)P(z)-P(t)P^*(z)]/(z-t)$ with $|P^*(t)|>|P(t)|$ has one zero less than $P(z)$ at points z for which $\Re(z)\Re(t)>0$ and has the same number of zeros as $P(z)$ at points z for which $\Re(z)\Re(t)<0$. This theorem is a generalization of I. Schur's result [Z. Angew. Math. Mech. 1, 307–311 (1921)] that, if $\Re(t)<0$, then $P(z)$ has all its zeros in the half-plane $\Re(z)<0$ if and only if the same is true of $P_1(z)$. This theorem together with the transformation $w=(z+\bar{t})/(z-t)$ where $\Re(t)<0$ is used to carry over to functions $f(z)$ bounded and analytic in $\Re(z)>0$ certain results of I. Schur [J. Reine Angew. Math. **147**, 205–232 (1917); **148**, 122–145 (1918)] on the continued fraction expansions of functions bounded and analytic in the unit circle. Also the results of H. S. Wall [Bull. Amer. Math. Soc. **52**, 138–143 (1946); these Rev. 7, 423] are extended to functions analytic in $\Re(z)>0$. [Reviewer's note. The first-mentioned theorem is also stated and proved by S. Benjaminowitsch, Monatsh. Math. Phys. **42**, 279–308 (1935).] *M. Marden* (Milwaukee, Wis.).

Sz. Nagy, Gyula. Die Lage der A-Stellen eines Polynoms bezüglich seiner Nullstellen. Acta Univ. Szeged. Sect. Sci. Math. **11**, 147–151 (1947).

The following is the principal result of this paper. Let $r_1,\cdots,r_n$ be any positive numbers and A any complex number such that $r_1r_2\cdots r_n=|A|$. Let C_j be the circles $|z-z_j|=r_j$, $j=1,\cdots,n$, and let $f(z)=(z-z_1)(z-z_2)\cdots(z-z_n)$. Then among the A-points of $f(z)$ (the points where $f(z)=A$), none lies either inside or outside all the C_j. If $|B|\leqq|A|$, all the B-points also lie in the circles C_j and precisely p of the B-points $(1\leqq p\leqq n)$ lie in any point set consisting of p circles C_j and having no point in common with the remaining circles C_j. An elementary proof is given for each part of the theorem. *M. Marden* (Milwaukee, Wis.).

Nikolsky, S. Sur la meilleure approximation d'une fonction dont la dérivée d'ordre s possède des discontinuités de la première espèce. C. R. (Doklady) Acad. Sci. URSS (N.S.) **55**, 95–98 (1947).

Soit $f(x)$ une fonction réelle définie dans $-1\leqq x\leqq 1$, ayant une dérivée d'ordre $s-1$ qui est la primitive d'une fonction $\varphi(x)$ finie, n'ayant que des discontinuités de première espèce dans $-1\leqq x\leqq 1$, mais en ayant au moins une dans $-1<x<1$. On désigne par $E_n(f)$ la meilleure approximation de f par un polynome de degré n, et on pose:

$$H=\max_{-1<x<1}|\varphi(x+0)-\varphi(x-0)|(1-x^2)^{\frac{1}{2}};$$

$$\mu(s)=\lim_{n\to\infty}n^sE_n(|x|^s);$$

on a alors, pour s impair, l'égalité asymptotique:

$$E_n(f)\sim\tfrac{1}{2}H\mu(s)n^{-s}/s!.$$

Si $|\varphi(x)|<K$, alors $H\leqq 2K$, mais si $\varphi(x)$ possède des discontinuités de 2ᵉ espèce, l'égalité asymptotique ci-dessus n'a pas lieu en général. *J. Favard* (Paris).

Nikolsky, S. Sur la meilleure approximation en moyenne par polynômes des fonctions ayant des singularités de la forme $|a-x|^s$. C. R. (Doklady) Acad. Sci. URSS (N.S.) **55**, 191–194 (1947).

On appelle meilleure approximation en moyenne, avec le poids $r(x)$, de la fonction $f(x)$ définie dans $-1\leqq x\leqq 1$, par un polynome de degré n, l'expression:

$$E_{n,r(x)}(f)=\min_{P_n}\int_{-1}^{1}|f(x)-P_n(x)|r(x)dx$$

(on pose $E_{n,1}(f)=E_n(f)$), le minimum étant à prendre parmi tous les polynomes P_n de degré n au plus. Lorsque le poids satisfait à certaines conditions, l'auteur donne l'expression asymptotique de $E_{n,r}(f)$ pour $f(x)=\sum_{k=1}^{m}A_k|x-a_k|^s$ $(-1<a_k<1; s>-1$ mais différent d'un nombre pair). Cette

expression reste valable pour $m=\infty$ lorsque $r=1$, pourvu que: $\sum_1^\infty |A_k| < \infty$. Lorsque s est un entier impair, et lorsque $f(x)$ possède dans $-1\leqq x\leqq 1$ une dérivée d'ordre $(s-1)$ qui est la primitive d'une fonction $\varphi(x)=g(x)+h(x)$ à variation bornée, où $g(x)$ est la fonction des sauts et où $h(x)$ est absolument continue, on en déduit également l'expression asymptotique de $E_n(f)$ lorsque φ a au moins une discontinuité dans $-1<x<1$. *J. Favard* (Paris).

Sapogov, N. Meilleure approximation d'une fonction ayant une singularité critique réelle sur l'ellipse de convergence. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **10**, 463–468 (1946). (Russian. French summary)

Let $E_n[f(x)]$ denote the best approximation on $(-1, 1)$ to $f(x)$ by polynomials $P_n(x)$ of degree not exceeding n, that is, the minimum of $\max |P_n(x)-f(x)|$ for $|x|\leqq 1$. If $t(x)$ is a positive continuous function, $E_{n,t(x)}[f(x)]$ denotes the minimum of $\max |P_n(x)-f(x)|t(x)$. It was shown by S. Bernstein [On Orthogonal Polynomials in a Finite Interval, Kharkov, 1937, p. 117] that for positive integral s and $a>1$,

$$E_{n,t(x)}[(a-x)^{-s}]\sim E_n[(a-x)^{-s}] \times\exp\left\{\frac{(a^2-1)^{\frac12}}{\pi}\int_{-1}^{1}\frac{\log t(x)\,dx}{(a-x)(1-x^2)^{\frac12}}\right\}.$$

The author extends the formula to arbitrary real values of s. He first rederives Bernstein's asymptotic form for $E_n[(a-x)^s]$ by differentiation and integration (in general, of fractional order) applied to a formula of Bernstein for $E_n[(a-x)^{-1}]$ [Extremal Properties of Polynomials and Best Approximation of Continuous Functions of a Single Real Variable, Leningrad-Moscow, 1937, p. 82]. Then he applies the same reasoning to the corresponding formula for $E_{n,t(x)}[(a-x)^{-1}]$ [first reference above, p. 116]. By comparing the two asymptotic formulas, he arrives at his result. The reader is assumed to be familiar with the two books of Bernstein cited above; formulas are quoted from them without explanation of the notation involved.

R. P. Boas, Jr. (Providence, R. I.).

Herpin, André. Sur une nouvelle méthode d'introduction des polynomes de Lucas. C. R. Acad. Sci. Paris **225**, 17–19 (1947).

It is shown that the general two-rowed square matrix F with complex elements is expressible as a linear combination of the unit matrix and the matrices $\sigma_1, \sigma_2, \sigma_3$ of Pauli. A formula is then obtained for the nth power F^n of F, which involves the Lucas polynomials U_n and V_n. This explains why the electrospherical polynomials of Aubert and Guillet, which are a special case of the Lucas polynomials, occur in problems of propagation involving the iteration of identical elements [see M. Parodi, Memor. Sci. Phys., no. 47, Gauthier-Villars, Paris, 1944; these Rev. 7, 295].

O. Frink (State College, Pa.).

Special Functions

Baudoux, P. Sur les équations du type de Bessel avec second membre. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **31** (1945), 471–478 (1946).

The purpose of this paper is to study the inhomogeneous Bessel equation (1) $y''+x^{-1}y'+(1-n^2/x^2)y=\phi(x)$, "in a general manner," by means of the Heaviside calculus. For $f(p)=p\int_0^\infty e^{-px}x^n y\,dx$ one has the differential equation $(p^2+1)f'+(2np-1/p)f=\Psi$ in which $\Psi(p)=-p\int_0^\infty e^{-px}x^{n+1}\phi\,dx$. By using this equation the author finds that for $\phi(x)=-x^{-n-1}\cos x$ a particular solution of (1) is $(2n-1)^{-1}x^{-n}\sin x$ and for $\phi(x)=-x^{-2}J_{n-1}(x)$ a particular solution is $2^{n-1}\pi^{-\frac12}(n-\frac12)!J_{n-1}(x)$ [in the second result the numerical factor seems to be incorrect]. Again, by introducing $(\frac12 x)^2$ as a new variable and taking Laplace transforms in a similar manner as before he obtains particular solutions of (1) for the following $\phi(x)$: (i) $x^{-2}J_{n-2}(x)$, (ii) $x^{-n}J_{n-4}(x)$, (iii) x^{2m+n}. Neither the results nor the method appear to be essentially new. *A. Erdélyi.*

Baudoux, P. Sur les fonctions de Weber et Lommel. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **31** (1945), 669–681 (1946).

In this paper the author studies the Weber-Lommel function $\Omega_n(x)$ by means of the Heaviside calculus. Some of his results are known [and recorded in G. N. Watson's Treatise on the Theory of Bessel Functions, Cambridge University Press, 1922, where the notation is $\mathbf{E}_n(x)$]; of the new results of interest mention should be made of integrals such as

$$\Omega_0(x)=-(2/\pi)\int_0^\infty J_0(2(sx)^{\frac12})\frac{\partial J_0(s)}{\partial n}ds$$

and several others, and of the expansion

$$\Omega_0(x)=2\pi^{-\frac12}\sum_{n=0}^{\infty}\frac{(\frac12 x)^{n+\frac12}}{n!(2n+1)}J_{n+\frac12}(x)$$

with a similar expansion for Ω_1. *A. Erdélyi.*

Truesdell, C. A note on the Poisson-Charlier functions. Ann. Math. Statistics **18**, 450–454 (1947).

The Poisson-Charlier polynomials p and functions ψ are defined by the relations

$$\psi_n(m, z)=\frac{(-)^m}{m!}\frac{d^n}{dz^n}(e^{-z}z^m),\qquad p_n(m, z)=\psi_n(m, z)/\psi_0(m, z).$$

The author suggests that these functions are most directly and easily studied in connection with his "F-equation" [Proc. Nat. Acad. Sci. U. S. A. **33**, 82–93 (1947); these Rev. **8**, 517]. He extends the definition of the Poisson-Charlier functions to arbitrary values of m, n, and proceeds to develop many properties of the generalised functions from his results on the F-equation. Finally he points out that

$$p_\beta(\alpha, z)=\cos(\beta-\alpha)\pi\cdot\Gamma(\alpha+1)z^{-\alpha}L_\alpha^{(\beta-\alpha)}(z),$$

so that the properties of the Poisson-Charlier functions can also be derived from the properties of generalised Laguerre functions [E. Pinney, J. Math. Phys. Mass. Inst. Tech. **25**, 49–79 (1946); these Rev. **7**, 442]. *A. Erdélyi.*

Kourganoff, Vladimir. Sur les intégrales $\int_0^\infty e^{-px}x^sK_n(x)dx$. C. R. Acad. Sci. Paris **225**, 430–431 (1947).

The integral referred to in the title is denoted by $I_{psn}(a)$. In this integral

$$K_n(x)=\int_1^\infty t^{-n}e^{-xt}dt,\qquad n=0, 1, 2, \cdots.$$

Replacement of K_n by this integral representation and change of the order of integrations leads to the evaluation in the form $I_{psn}(a)=a^{-s-1}(s+n)^{-1}s!F(s+1, s+n; s+n+1; -p/a)$. For numerical computations it is, however, more conven-

ient to reduce I_{psn} by integrations by parts to $I_{p,0,n+s}$: this last integral can then be evaluated in finite form. A more detailed study of K_n and of I_{psn} will appear elsewhere.

A. Erdélyi (Pasadena, Calif.).

Kourganoff, Vladimir. Sur les intégrales

$$I_{psnm}(a, b)=\int_0^\infty e^{-px}x^s K_n(ax)K_m(bx)dx.$$

C. R. Acad. Sci. Paris **225**, 451–453 (1947).

[Cf. also the preceding review.] The integral is transformed into a triple integral by substitution of the integral representation of K_n. When $p=s=0$, the evaluation of I in finite form follows at once; $I_{p011}=V_p$ can be expressed in terms of Euler's dilogarithm, I_{p0n1} in terms of V_p and of elementary functions, I_{p0nm} can be computed by means of recurrence relations from the integrals already evaluated, and similarly I_{psnm}. Numerical values of $I_{10nm}(1, 1)$ are given for $n, m=1, 2, 3, 4$. These integrals too will be discussed in more detail elsewhere. *A. Erdélyi* (Pasadena, Calif.).

Bailey, W. N. Well-poised basic hypergeometric series. Quart. J. Math., Oxford Ser. **18**, 157–166 (1947).

Relations between the well-poised basic hypergeometric series ${}_{10}\Phi_9$ are developed analogous to those previously presented by the author for the well-poised ${}_9F_8$ [Generalized Hypergeometric Series, Cambridge University Press, 1935, pp. 51, 64]. *N. A. Hall* (Minneapolis, Minn.).

Differential Equations

Bajada, E. Le approssimazioni nella risoluzione delle equazioni differenziali ordinarie. I. Teorema d'esistenza. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **2**, 261–268 (1947).

Bajada, E. Le approssimazioni nella risoluzione delle equazioni differenziali ordinarie. II. Applicazioni. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **2**, 398–402 (1947).

The differential equation $y'=f(x, y)$ with $y(x_0)=y_0$ is considered in a rectangle R of the (x, y)-plane with center at (x_0, y_0). Let $f(x, y)$ be continuous in R. A successive approximation procedure due to Tonelli yields a sequence of functions with exactly one function of accumulation which is a solution of $y'=f(x, y)$, $y(x_0)=y_0$. In general there are an infinite number of such solutions and the author generalizes Tonelli's procedure so as to get an infinite variety of approximation procedures. For any given solution of the problem the author shows that at least one of his procedures will yield a sequence which has this solution as a function of accumulation. In the second paper the author applies his theorem to $y'=|y|^{\frac{1}{2}}$. *N. Levinson* (Cambridge, Mass.).

Rosenblatt, Alfred. On E. Picard's method of successive approximations in the case of a system of two ordinary differential equations of the first order. Revista Ci., Lima **49**, 167–178 (1947). (Spanish)

Picard's method of successive approximations is used to prove the existence and uniqueness of a solution of the system of differential equations and initial conditions $y'=f(x, y, z)$, $z'=g(x, y, z)$, $y(0)=z(0)=0$, in the case in which the functions f and g satisfy conditions of the forms

$$|f(x, y_1, z_1)-f(x, y_2, z_2)|\leqq A|z_1-z_2|/x^{m/p},$$
$$|g(x, y_1, z_1)-g(x, y_2, z_2)|\leqq B|y_1-y_2|/x^{n/p}.$$

Here m, n and p are positive integers satisfying the conditions $m+n=2p$, $\bar{A}\bar{B}=n/p<1$, $np^{-1}+np^{-1}A/\bar{A}+B/\bar{B}<1$. The symbols $\bar{A}$, $\bar{B}$ are not defined. They do not appear again in the paper, and their significance remains obscure. Typographical errors make the correct interpretation of sundry other details uncertain. *L. A. MacColl.*

Hartman, Philip. The L^2-solutions of linear differential equations of second order. Duke Math. J. **14**, 323–326 (1947).

Let $f(t)$ and $g(t)$ be real-valued and continuous for $0\leqq t<\infty$ and let $f(t)$ be bounded from above and $g(t)$ be in $L^2(0, \infty)$. Let $x(t)$ be in $L^2(0, \infty)$ and a solution of $x''+f(t)x=g(t)$. Then $x'(t)\epsilon L^2(0, \infty)$ and $x'(t)\rightarrow 0$ as $t\rightarrow\infty$. This answers a question raised by Wintner.

N. Levinson (Cambridge, Mass.).

Wintner, A. Unrestricted Riccatian solution fields. Quart. J. Math., Oxford Ser. **18**, 65–71 (1947).

The author proves the following theorem. On the half-plane $x\geqq 0$ let $f(x, y)$ be a real-valued continuous function which changes its sign exactly twice on every line $-\infty<y<\infty$, $x=x$, say at $y=y^-(x)$ and $y=y^+(x)$, where $y^-(x)<y^+(x)$, $0\leqq x<\infty$. Suppose $f(x, y)$ is positive between its zeros (otherwise change y to $-y$) and let $y^-(0)\geqq y^-(x)$ and $y^+(0)\leqq y^+(x)$, $0\leqq x<\infty$. Then every solution $y(x)$ of $y'=f(x, y)$ defined for small x and with $y(0)>y^-(0)$ exists on the whole half-line $x\geqq 0$. By further restricting the hypothesis the author obtains information about the asymptotic behavior of $y(x)$ in two further theorems.

N. Levinson (Cambridge, Mass.).

Haag, Jules. Sur certains systèmes d'équations différentielles à solutions périodiques. Bull. Sci. Math. (2) **70**, 155–172 (1946).

The author extends the results of Poincaré on the existence and stability of periodic solutions of ordinary differential equations to certain cases where the functions involved are not analytic. He considers systems of the form $dx_i/dt=\lambda f_i(x_1, \cdots, x_k, t)$, where the f_i are periodic in t, $\lambda>0$ is a small parameter, the partial derivatives $\partial f_i/\partial x_j$ exist, are continuous and satisfy a Hölder condition. In the proof, the case where f_i is linear and homogeneous in $x_1, \cdots, x_k$ is first treated by the standard procedures; the case where f_i is linear but not homogeneous is then treated by a variation of constants; and, finally, the general case is dealt with by the method of successive approximations. *P. Hartman.*

Hukuhara, Masuo. Sur les points singuliers des équations différentielles linéaires. III. Mem. Fac. Sci. Kyûsyû Imp. Univ. A. **2**, 125–137 (1942).

[For part II cf. the same Mem. A. **2**, 1–25 (1941); these Rev. **3**, 120.] The paper is concerned with the asymptotic representation of solutions of a system

$$(1)\qquad \frac{dy_j}{dx}=\lambda_j(x)y_j+x^{-1}\left\{\sum_{k=1}^{n}a_{j,k}(x)y_k+a_j(x)\right\},\qquad j=1, \cdots, n,$$

in a domain (2) $\theta_1<\arg x<\theta_2$, $|x|>R$. The coefficients $\lambda_j(x)$ are polynomials of any degree and the $a_{jk}(x)$ and $a_j(x)$ are analytic in (2) and asymptotically developable in series in negative powers of x in the region (3) $\theta_1\leqq\arg x\leqq\theta_2$, $|x|\geqq R$.

Relative to each of the polynomials $\Lambda_j(x)$ $(=\int_0^x\lambda_j(x)dx)$ the exterior of a circle $|x|=R$ is subdivided into regions in which $\Re(\Lambda_j(x))$ is alternately negative and nonnegative. If no two such regions in which $\Re(\Lambda_j(x))\geqq 0$, for any j, are

joined by a domain $\theta_1-\epsilon\leqq\arg x\leqq\theta_2+\epsilon$, $|x|\geqq R_0$, with R_0 sufficiently large and ϵ sufficiently small but positive, the region (3) is said to be proper. The principal theorem is the following one. If the region (3) is proper, there exist solutions of (1) that are asymptotically developable in series $y_j(x)\sim\sum_{m=0}^{\infty}\alpha_j^{(m)}/x^m$ in the region. Their number is equal to one plus the number of indices j for which the region (3) is contained in a domain in which $\Re(\Lambda_j(x))<0$.

The author also discusses the reduction of the system (1) to canonical form and contrasts his results with certain related ones of M. Malmquist. *R. E. Langer.*

Cuénod, Michel. Étude des propriétés d'un réglage automatique. Essai de synthèse de différentes méthodes de calcul. Application au réglage de vitesse d'un groupe hydro-électrique. Bull. Tech. Suisse Romande **73**, 105–115, 121–125 (1947).

This is an expository article on the methods of Hurwitz, Leonhard, Küpfmüller and Nyquist for determining the stability of a servomechanism. The methods are compared in application to a servomechanism controlling a turbo-generator. *M. Marden* (Milwaukee, Wis.).

Müller-Strobel, Josef. Störungstheorie und Stabilität. Anwendung der Störungstheorie zur näherungsweisen Bestimmung der statischen Stabilitätsgrenzen von Synchronmaschinen in vermaschten Netzen. Arch. Elektrotechnik **37**, 555–587 (1943). [MF 15625]

The author considers the stability of an arbitrary number of synchronous machines coupled to a power distributive network, represented by a system of second order differential equations of motion, with coefficients dependent upon the conductances, voltages, phase angles, angular velocities and power outputs of the elementary machines. In an earlier paper [same Arch. **36**, 573–584 (1942)] the author obtained stability criteria through conditions imposed upon only the (in this case, power) coefficients of the restorative terms. In a later paper [same Arch. **37**, 509–519 (1943)] the equations are discussed by small amplitude procedures, assuming constant network conductances and linear relationships between power, voltage and angular velocity, obtaining as condition of stability that the Hurwitz determinant is satisfied. In the present study the author observes that this leads to difficult calculations and re-examines the validity and degree of approximation of the stability criteria. Assuming the variation of the power output and of the induced voltages of the machines to approach zero, while the variation of phases and of network conductances relative to phase are different from zero, the equations of motion are discussed along classical Lagrangian lines, it being assumed that the eigenvalues are distinct among elementary machines. Following the standard perturbation procedure, the characteristic determinant of the motional equations is developed into a summed sequence of product terms and the degree of approximation in the resulting stability criteria discussed for various choices of terms used. A first approximation gives stability criteria corresponding to those of the author's earlier work, while closer approximation leads to different criteria. Two numerical examples are given.
J. H. Bigelow (Princeton, N. J.).

Debever, R. Quelques problèmes d'équivalence de formes différentielles alternées. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **31** (1945), 262–277 (1946).

Two symbolic (alternées) differential forms, one in variables x and the other in variables X, are said to be equivalent if there exists a nonsingular transformation $x\leftrightarrow X$ which transforms one form to the other. Let ω^i and Ω^i be symbolic linear forms in the variables x and X, respectively. The paper shows that the quadratic forms in four variables $\omega=\omega^1\omega^2+\omega^3\omega^4$ and $\Omega=\Omega^1\Omega^2+\Omega^3\Omega^4$ are always equivalent if they both have rank four. The question of sending ω and $\pi=\omega^1\omega^2-\omega^3\omega^4$ into Ω and $\Pi=\Omega^1\Omega^2-\Omega^3\Omega^4$ by a single transformation $x\leftrightarrow X$ is also discussed. *F. G. Dressel.*

Saltykow, N. Forme générale des équations différentielles de dynamique, à deux paramètres, intégrables par séparation des variables. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **32** (1946), 576–587 (1947).

In order to solve the problem indicated in the title, the author considers the characteristic function

$$H\equiv Ap_1^2+2Bp_1p_2+Cp_2^2+2Dp_1+2Ep_2+U,$$

where $p_i=\partial z/\partial x_i$ and the coefficients are functions of x_1 and x_2. He then calculates the values of these functions in order that z be obtainable by quadratures. The final result is

$$z=\int\frac{C_1+C_2Y_1-V_1}{X_1}dx_1+\int\frac{C_1-C_2Y_2-V_2}{X_2}dx_2+C_3,$$

where C_1, C_2, C_3 are arbitrary constants, C_1 and C_2 being dependent, and the functions involved contain only the variables indicated by the subscripts. *M. S. Knebelman.*

Stojadinovitch, M. Équations aux dérivées partielles de dynamique, à trois paramètres, intégrables par séparation des variables. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **32** (1946), 588–591 (1947).

The same method as in the paper reviewed above is applied to the characteristic function

$$H\equiv A\mu_1^2+2B\mu_1\mu_2+C\mu_2^2+2D\mu_1\mu_3+2E\mu_2\mu_3+G\mu_3^2,\qquad \mu_i=\partial z/\partial x_i,\quad i=1,2,3.$$

There are eight cases where integration by separation of variables is possible and in this paper the author gives the solution for the first four. In each case z is expressed in terms of three integrals involving three arbitrary constants.
M. S. Knebelman (Pullman, Wash.).

Wischik, M. J. La méthode des projections orthogonales pour les équations différentielles conjuguées à elles-mêmes. C. R. (Doklady) Acad. Sci. URSS (N.S.) **56**, 115–118 (1947).

The "orthogonal projection" method for finding the solution of a partial differential equation satisfying certain "boundary conditions" consists in constructing a Hilbert space H and a linear subspace L of H such that, if ψ is the element of H which gives the boundary conditions, then the projection of ψ into L gives the solution of the boundary problem. The method had been used by H. Weyl [Duke Math. J. **7**, 411–444 (1940); these Rev. **2**, 202] for the Laplace equation. In the present note it is applied to an equation of the form $u_{xx}+u_{yy}-(q-a_x-b_y)u=0$ with $q-a^2-b^2>0$ under certain differentiability conditions for the functions q, a and b of x, y. (The author states that it is easy to extend his results to the case of more than two dimensions.) The so-called first, second and third boundary value problems are treated. The term "boundary value problem" for the domain G in the (x, y)-plane given by the element $\psi(x, y)=(\psi_1(x, y), \psi_2(x, y), \psi_3(x, y))$ is, e.g. in the third case, to be understood in the following sense: if

G_i ($i=1, 2, \cdots$) is a sequence of domains such that $\lim_{i\to\infty} G_i=G$ and such that for the boundary Γ_i of each G_i a normal n exists, then a solution u of the boundary value problem is a function u satisfying in addition to the above partial differential equation the "boundary condition" that the weak limit of

$$\partial u/\partial n+u(a\cos(nx)+b\cos(ny)) \\ -\{\psi_1\cos(nx)+\psi_2\cos(ny)+\psi_3(a\cos(nx)+b\cos(ny))\},$$

taken on Γ_i, as $i\to\infty$ equals 0. *E. H. Rothe.*

Ghizzetti, Aldo. Sul metodo della trasformata parziale di Laplace a intervallo di integrazione finito. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) 6, 1–47 (1947).

Let T be a region in the (x, y)-plane bounded by the curves $y=\alpha(x)$, $y=\beta(x)$ and the ordinates $x=0$ and $x=1$. The author considers the following three problems: (a) the Dirichlet problem for the equation $u_{xx}+u_{yy}-\lambda^2u=F(x, y)$ for the region T; (b) the Neumann problem for the same equation and region; (c) the equilibrium problem for a plate with given load over T and clamped at the boundary. Following Picone [Ann. Sci. Univ. Jassy 26, 183–232 (1940); these Rev. 1, 236] and Amerio [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 1, 346–352, 544–548 (1946); these Rev. 8, 274] it is shown that a formal application of the Laplace transformation with respect to y over the finite interval $\alpha(x)<y<\beta(x)$ will reduce problems (a), (b), (c) to infinite systems of integral equations of the Fischer-Riesz type for u and its derivatives on the arcs $y=\alpha(x)$ and $y=\beta(x)$. The author also gives direct rigorous proofs for the equivalence of the problems (a), (b), (c) with the corresponding integral equations. These proofs make use of Green's formula and of explicit expressions for the Green's function for the strip $0<x<1$. The same Green's functions can be used to determine u in the interior of T, once u and its derivatives are known on the arcs $y=\alpha(x)$ and $y=\beta(x)$. *F. John* (New York, N. Y.).

Ghizzetti, A. Sul metodo della trasformata parziale di Laplace a intervallo di integrazione finito. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 1, 691–696 (1946).

A note containing the results, without proofs, of the paper reviewed above. *F. Zernike* (Baltimore, Md.).

Amerio, Luigi. Sull'integrazione dell'equazione $\Delta_{2k}u=f$. Ann. Mat. Pura Appl. (4) 24, 119–138 (1945).

Let u be a solution of the equation $\Delta_{2k}u=f$ in a region τ of m dimensional space with boundary σ. By means of Green's formula, it is possible, assuming suitable smoothness of u and σ, to obtain an integral representation of u in terms of f and the boundary values of $\Delta_{2h}u$ and $\partial(\Delta_{2h}u)/\partial n$, $h=0, 1, \cdots, k-1$. However, not all these functions can be assigned on σ, and Green's formula gives an integral identity relating them. Furthermore, if functions satisfying this integral identity are used in the representation formula, the resulting function u satisfies $\Delta_{2k}u=f$ and u and its appropriate derivatives assume the functions as boundary values. The author studies the integral identity, and others derived from it by differentiating with respect to parameters, by considering them as orthogonality conditions imposed on the boundary values. If a suitable independent set of boundary values are selected, the others are determined by expansions in series of orthogonal functions. By considering the Laplace transform of u, he is able to reduce the expansions to the case of series of orthogonal polynomials satisfying $\Delta_{2k}u=0$. *J. W. Green* (Los Angeles, Calif.).

Vodička, Václav. Le mouvement de la chaleur dans une matière liquide infusée dans un vase sphérique. Věstník Královské České Společnosti Nauk. Třída Matemat.-Přírodověd. 1946, no. 11, 8 pp. (1947). (Czech. French summary)

The problem is that of conduction of heat in a hollow sphere (of radii $R>r$) filled with a liquid. The author uses spherical polar coordinates ρ, θ, φ and considers symmetrical temperature distributions (that is, distributions independent of θ and φ). There are two different partial differential equations, one valid in the liquid ($0\leq\rho<r$) and one in the shell ($r<\rho<R$), there is the condition of finiteness of the temperature at the centre ($\rho=0$), and finally there are the radiation conditions at the surfaces of transition, two (one into each of the two media) at the inner boundary ($\rho=r$), and one at the outer boundary ($\rho=R$) of the shell. The normal solutions can be expressed in terms of elementary functions, and the (transcendental) equation determining the characteristic values of the problem is written down. Thereupon the author proceeds to obtain the formulae for the solution with a given initial distribution of the temperature. The functions describing the initial distribution (one for each medium) are assumed to be continuous, differentiable and quadratically integrable. The author remarks that in spite of the elementary nature of the problem and the perfectly straightforward method of the solution the formulae become complicated on account of the presence of two differential equations in the two media. *A. Erdélyi.*

Mindlin, J. A. Problème mixte pour l'équation des ondes dans le cas d'un cercle et d'une sphère. C. R. (Doklady) Acad. Sci. URSS (N.S.) 56, 141–144 (1947).

The author solves the wave equation $u_{xx}+u_{yy}=a^{-2}u_{tt}$ for the "mixed" conditions (1) $u=u_0(r, \theta)$, $u_t=u_1(r, \theta)$ for $r<1$, $t=0$; (2) $u=\varphi(\theta, t)$ for $r=1$, $t>0$, in polar coordinates r, θ. The solution u is written in the form

$$u=\sum_{n=0}^{\infty}\int_0^{\infty}\{[A_n^{(1)}(at-r\cosh\xi)+A_n^{(2)}(at+r\cosh\xi)]\cos(n\theta) \\ +[B_n^{(1)}(at-r\cosh\xi)+B_n^{(2)}(at+r\cosh\xi)]\sin(n\theta)\}\cosh(n\xi)d\xi.$$

The initial conditions (1) lead to integral equations for the coefficients $A_n^{(i)}(z)$, $B_n^{(i)}(z)$ for $|z|<1$, which can be solved explicitly. The boundary condition (2) then permits one to find those coefficients for any z by solving a finite number (depending on z) of Volterra integral equations. Similar results are indicated for the 3-dimensional wave equation in the case of a sphere. *F. John* (New York, N. Y.).

Bureau, Florent. Sur l'intégration de l'équation des ondes. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 31 (1945), 610–624 (1946).

Bureau, Florent. Sur l'intégration de l'équation des ondes. II. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 31 (1945), 651–658 (1946).

Solution of the initial value problem for the n-dimensional inhomogeneous wave equation, using spherical means. The author's treatment and results are similar to the ones given by Courant and Hilbert [Methoden der mathematischen Physik, v. 2, Berlin, 1937, chap. 6; see, in particular, pp. 411–415]. *F. John* (New York, N. Y.).

Bureau, Florent. Sur l'intégration d'une équation linéaire aux dérivées partielles, totalement hyperbolique d'ordre quatre et à quatre variables indépendantes. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **33**, 185–199 (1947).

The author solves the Cauchy problem for the equation

$$(1)\qquad (a^2\partial^2/\partial t^2-\Delta)(b^2\partial^2/\partial t^2-\Delta)u=f(x_1, x_2, x_3, t),$$

where $\Delta=\partial^2/\partial x_1^2+\partial^2/\partial x_2^2+\partial^2/\partial x_3^2$, for any space-like initial surface S. Following Hadamard the author makes use of an "elementary" solution of the differential equation, which for equation (1) turns out to be the function

$$v(r, t)=\tfrac{1}{8}\pi^{-3}(b^2-a^2)^{-1}(v_b(r, t)-v_a(r, t)),$$

where r has an obvious meaning and the function $v_\lambda(r, t)$ is defined by

$$v_\lambda(r,t)=\begin{cases} 0, & t<\lambda r,\\ r^{-1}[(t-\lambda r)\log(t-\lambda r)-(t+\lambda r)\log(t+\lambda r)], & t>\lambda r.\end{cases}$$

Green's formula is applied to a conical region D, in which v is regular, and which tends towards the region bounded by S and the outer characteristic cone with vertex P. A careful passage to the limit yields an expression for $u(P)$ in terms of the initial data on S. *F. John.*

Lahaye, Edmond. La méthode de Riemann appliquée à la résolution d'une catégorie d'équations linéaires du troisième ordre. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **31** (1945), 479–494 (1946).

The method of Riemann is again shown to apply to the equation

$$(1)\qquad V_{xyz}+L_1V_{xy}+L_2V_{xz}+L_3V_{yz}+M_1V_x+M_2V_y+M_3V_z=F(x, y, z),$$

$$L_i=L_i(x, y, z),\quad M_i=M_i(x, y, z),\quad V_x=\partial V/\partial x,\quad i=1, 2, 3,$$

when the given data are on characteristic surfaces of (1) or on a noncharacteristic surface of (1). The work of L. Bianchi [Atti Accad. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (5) 4_1, 8–18, 89–99 (1895)] and O. Niccoletti [ibid., 330–337 (1895)] on this problem is not mentioned. *F. G. Dressel.*

Functional Analysis

Zitlanadze, E. S. Certains problèmes de l'extrêmé relatif et de la théorie des valeurs caractéristiques. C. R. (Doklady) Acad. Sci. URSS (N.S.) **56**, 15–18 (1947).

A continuation of earlier investigations of the author [same C. R. (N.S.) **53**, 307–309 (1946); these Rev. **8**, 386]. Theorem 1 (dealing with the Lusternik-Schnirelmann "critical point principle") is an extension of theorem 1 of the previous paper from Hilbert to Banach spaces. Theorem 2 states that a "symmetric" (not necessarily linear) operator $L(a)$ in a separable Hilbert space is totally continuous if and only if the functional from which it is derived is weakly continuous. Theorem 3 states the existence of at least one eigenvalue of such an operator. An existence theorem of Lichtenstein [Vorlesungen über einige Klassen nichtlinearer Integralgleichungen und Integro-differentialgleichungen nebst Anwendungen, Springer, Berlin, 1931, p. 141] is quoted as a special case of theorem 3. Not all proofs are given in detail. *E. H. Rothe* (Ann Arbor, Mich.).

Pinsker, A. G. On concrete representations of linear semi-ordered spaces. C. R. (Doklady) Acad. Sci. URSS (N.S.) **55**, 379–381 (1947).

By using Kakutani's theorem on the representation of abstract L spaces the author improves one of his own representation theorems which had previously been demonstrated only for separable spaces. The general result may be stated as follows. Let X be an arbitrary K-space and $X=\sum X_\alpha$ ($\alpha\epsilon A$) its decomposition into a disjunctive sum of elementary subspaces. Then for every $\alpha\epsilon A$ there exists a bicompact Hausdorff space Ω_α with a completely additive measure $m(E)$, defined for all Borel sets $E\subset\Omega_\alpha$, such that X_α is normally contained in the space of measurable functions $S(\Omega_\alpha; m)$ defined in Ω_α. If the symbol Ω denotes the set-theoretical sum of spaces Ω_α, and $S(\Omega; m)$ is the space of functions defined in Ω and measurable on every Ω_α, then X is isomorphic to the normal subspace of the space $S(\Omega; m)$. Moreover, the maximal extension of X is isomorphic to $S(\Omega; m)$, $\hat{X}=S(\Omega; m)$, and to the unity of X corresponds the unity of $S(\Omega; m)$. *N. Dunford* (New Haven, Conn.).

Hewitt, Edwin. Certain generalizations of the Weierstrass approximation theorem. Duke Math. J. **14**, 419–427 (1947).

Soit $\mathfrak{C}(X)$ l'ensemble des fonctions continues et bornées à valeurs réelles, définies sur un espace topologique X. Un ensemble $\mathfrak{D}$ de fonctions appartenant à $\mathfrak{C}(X)$ est dit "ensemble de générateurs analytiques" pour $\mathfrak{C}(X)$, si toute fonction appartenant à $\mathfrak{C}(X)$ est limite, uniforme sur tout l'espace X, d'une suite de fonctions, qui sont des polynômes à coefficients réels par rapport à des fonctions appartenant à $\mathfrak{D}$. Deux ensembles F_1, F_2 de X sont dits séparés par une fonction $\varphi\epsilon\mathfrak{C}(X)$ si les valeurs de φ sur F_1 et les valeurs de φ sur F_2 sont respectivement contenues dans deux intervalles fermés sans point commun.

Un théorème de M. H. Stone [Trans. Amer. Math. Soc. **41**, 375–481 (1937), théorème 82] exprime que, si X est compact, pour que $\mathfrak{D}$ soit un ensemble de générateurs analytiques de $\mathfrak{C}(X)$, il faut et il suffit que deux points quelconques distincts de X puissent être séparés par une fonction $\varphi\epsilon\mathfrak{D}$. L'auteur montre que cette condition n'est jamais suffisante, si X n'est pas compact. Il donne la condition suffisante suivante: si X est complètement régulier (c'est-à-dire si tout ensemble fermé et tout point de X ne lui appartenant pas peuvent être séparés par une fonction $f\epsilon\mathfrak{C}(X)$) $\mathfrak{D}$ sera un ensemble de générateurs analytiques de $\mathfrak{C}(X)$ si deux ensembles fermés quelconques, F_1, F_2, pouvant être séparés par une fonction $\varphi\epsilon\mathfrak{C}(X)$, peuvent aussi être séparés par une fonction $\psi\epsilon\mathfrak{D}$. Cette condition est équivalente à celle de Stone si X est compact. Diverses variantes et conséquences sont données. *L. Schwartz.*

Dunford, Nelson, and Hille, Einar. The differentiability and uniqueness of continuous solutions of addition formulas. Bull. Amer. Math. Soc. **53**, 799–805 (1947).

The study of a one parameter semi-group T_ξ ($0\leqq\xi<\infty$) of bounded linear operators on a Banach space Ξ which satisfies $T_{\xi+\zeta}=T_\xi T_\zeta$ leads to consideration of the addition law $f(\xi+\zeta)=f(\xi)f(\zeta)$, where $f(\xi)$ is a function on $0\leqq\xi<\infty$ to a commutative complex Banach algebra (with unit) of bounded linear operators on Ξ. This leads the authors to consider the more general case: (1) $f(\xi+\zeta)=G[f(\xi), f(\zeta)]$, $0\leqq\xi, \zeta, \xi+\zeta\leqq\omega$, where $f(\xi)$ is a function on $0\leqq\xi\leqq\omega$ to a commutative complex Banach algebra B with unit e. Here $G[\alpha, \beta]$ is a symmetric complex function for α, β in the

closure of a domain Δ (of the complex plane) bounded by a rectifiable Jordan curve; $G[u, v]$ is defined only for $u, v\epsilon B(\Delta)$, the subset of B of elements x whose spectrum $\sigma x \subseteq \Delta$. For such u, v,

$$G[u, v] = (2\pi i)^{-2}\int_{\Gamma_u}\int_{\Gamma_v} G[\alpha, \beta]R(\alpha, u)R(\beta, v)d\alpha d\beta,$$

where $R(\alpha, u) = (\alpha e - u)^{-1}$ and Γ_u, Γ_v are oriented envelopes in Δ of σu, σv, respectively. Results are obtained on the uniqueness and differentiability of solutions of (1).

R. A. Leibler (Princeton, N. J.).

Arnous, Edmond. Méthodes d'approximation: La condition d'extremum $\delta \frac{\int \bar{\psi} H \psi d\tau}{\int \bar{\psi}\psi d\tau} = 0$ et l'écart type de la loi de probabilité de H. C. R. Acad. Sci. Paris **225**, 449–451 (1947).

Let H be a Hamiltonian operator on Hilbert space, $m_1(\psi) = (\psi, H\psi)/(\psi, \psi)$ and $\sigma^2(\psi) = \| (H - m_1)\psi/\|\psi\| \|^2$. Then using an argument involving the proper functions of H it can be shown that $\delta m_1(\psi) = 0$ if and only if $\sigma(\psi) = 0$. The author obtains this result without recourse to proper functions. [The result also holds if it is only assumed that H is Hermitian.] *R. A. Leibler* (Princeton, N. J.).

Alexiewicz, A. Linear operations among bounded measurable functions. I. Ann. Soc. Polon. Math. **19** (1946), 140–160 (1947).

Let M be the space of real functions, defined over a set E, essentially bounded with respect to a completely additive measure μ. The author proves representation theorems for continuous linear operators from M to a linear space Y where various sequential limit topologies are imposed on both M and Y. Such operators are called (M_γ, Y_δ) linear, where the γ and δ refer to the type of topology on M and Y and their representations are given in terms of Fréchet-Radon integrals $\int x(t)d\Phi$, where Φ is an additive set function with values in Y. Necessary and sufficient conditions are given on Φ to insure continuity of the operator. In the latter part of the paper the author proves generalizations of the Banach-Steinhaus theorem on the continuity of a linear operator which is the limit of a sequence of continuous linear operators. The theorem is generalized to the case of operators linear (M_γ, Y_δ). *R. E. Fullerton* (Madison, Wis.).

Alexiewicz, A. Linear operations among bounded measurable functions. II. Ann. Soc. Polon. Math. **19** (1946), 161–164 (1947).

The generalizations of the Banach-Steinhaus theorem in the preceding paper are further generalized by taking as domain of definition of the operators the space $M\{X\}$ of essentially bounded functions with range in a Banach space X. *R. E. Fullerton* (Madison, Wis.).

Theory of Probability

Seal, H. L. A probability distribution of deaths at age x when policies are counted instead of lives. Skand. Aktuarietidskr. **30**, 18–43 (1947).

The author considers the distribution of policy claims, i.e., the distribution of the number of policies falling in by death. It is assumed (1) that the probability distribution of multiple policies is known, (2) that the probability distribution of deaths at any age x is a Poisson distribution. Formulae are derived for the distribution of policy claims and its characteristic function. The first seven ordinates and the first four moments of the distribution are obtained and asymptotic expressions for a large number of policies exposed to risk are derived. Numerical illustrations are also given. *E. Lukacs* (Cincinnati, Ohio).

Schutzenberger, Marcel-Paul. Sur certains paramètres caractéristiques des systèmes d'événements compatibles et dépendants et leur application au calcul des cumulants de la répétition. C. R. Acad. Sci. Paris **225**, 277–278 (1947).

Let $a_1, \cdots, a_n$ be events. The "deviation from independence" is defined as

$$\gamma(a_1, \cdots, a_n) = 2^n \sum (-1)^{\nu-1}(\nu-1)! \Pr(B_1) \cdots \Pr(B_\nu),$$

where the summation is extended over all partitions of the a_j's into ν groups $B_1, \cdots, B_\nu$, $\nu = 1, \cdots, n$. The function γ vanishes if the a_j's can be divided into two mutually independent groups of events; it changes sign if an a_j is replaced by its negation. It is shown how to obtain the semi-invariants of the number of times r that an event occurs in a succession of n not necessarily independent trials in terms of appropriate γ's, and thereby how to obtain theorems on the asymptotically Gaussian character of r. *J. L. Doob.*

Quenouille, M. H. On the problem of random flights. Proc. Cambridge Philos. Soc. **43**, 581–582 (1947).

Chandrasekhar [Rev. Modern Phys. **15**, 1–89 (1943); these Rev. **4**, 248] has described the treatment of random walk problems by means of Markov chains. For the case where all directions are equally likely he has also given a normal approximation to the probability distribution of the total displacement in n steps. The author finds an explicit formula and shows by numerical computations for $n = 10$ that Chandrasekhar's approximation is surprisingly good.

W. Feller (Ithaca, N. Y.).

Coulson, C. A. Note on the random-walk problem. Proc. Cambridge Philos. Soc. **43**, 583–586 (1947).

In a two-dimensional random walk all directions are supposed to be equally probable and the length of each step is a given function of the direction. The probability distribution of the total displacement in n steps is derived by the method of Markov chains as outlined by Chandrasekhar [Rev. Modern Phys. **15**, 1–89 (1943); these Rev. **4**, 248]. The same problem has been treated in a heuristic manner by Domb [same Proc. **42**, 245–249 (1946); these Rev. **8**, 281]. It turns out that the latter's approximations are justified only under certain restrictive conditions. *W. Feller.*

Chung, Kai Lai. On the maximum partial sum of independent random variables. Proc. Nat. Acad. Sci. U. S. A. **33**, 132–136 (1947).

Soit $\{X_n\}$ $(n = 1, 2, \cdots, \infty)$ une suite indéfinie de variables aléatoires indépendantes telles que: $E(X_n) = 0$, $E(X_n^2) = 1$, et posons: $S_n = \sum_{\nu=1}^n X_\nu$, $S_n^* = \max_{1 \leq \nu \leq n} |S_\nu|$. En supposant que les $E(|X_n|^3)$ soient bornés (mais cette restriction peut être affaiblie), l'auteur démontre que: la probabilité que

$$S_n^* < 8^{-\frac{1}{2}}\pi n^{-\frac{1}{2}}\{\log_2 n + 2\log_3 n + \log_4 n + \cdots + \log_{p-1} n + (1+\delta)\log_p n\}^{-\frac{1}{2}}$$

pour une infinité de valeurs de n est 0 ou 1 suivant que $\delta > 0$ ou $\delta < 0$. La démonstration est résumée, mais son principe est indiqué de façon assez détaillée. *R. Fortet.*

Kac, M., and Siegert, A. J. F. An explicit representation of a stationary Gaussian process. Ann. Math. Statistics **18**, 438–442 (1947).

Let $\rho(t)$, with $\rho(0)=1$, be the correlation function of a stationary Gaussian process. Let $\lambda_1, \lambda_2, \cdots$ be the characteristic values of the integral equation $\int_0^T\rho(s-t)f(t)dt=\lambda f(s)$ and let $f_1, f_2, \cdots$ be the corresponding normalized characteristic functions. The authors show that the series $x(t)=\sum_j\lambda_j^{\frac{1}{2}}G_jf_j(t)$, where $G_1, G_2, \cdots$ are mutually independent Gaussian random variables with zero means and unit variances, defines a Gaussian stochastic process in the interval $(0, T)$ with the given correlation function. The series converges with probability 1 for each t, and in the mean in t for almost all G_j's. Using this representation it is shown that the distribution of $T^{-\frac{1}{2}}[\int_0^Tx(t)^2dt-T]$ is asymptotically Gaussian with zero mean and variance $\int_{-\infty}^{\infty}\rho(t)^2dt$. [Cf. J. Appl. Phys. **18**, 383–397 (1947); these Rev. **8**, 522, for applications to the theory of noise.] *J. L. Doob.*

Kampé de Fériet, Joseph. Fonctions aléatoires définies sur un groupe abstrait. C. R. Acad. Sci. Paris **225**, 428–429 (1947).

Generalizing his earlier remarks [C. R. Acad. Sci. Paris **225**, 37–38 (1947); these Rev. **9**, 46], the author remarks that, if $f(\omega)$ is a measurable function on an abstract space of points ω, if the ω measure has value 1 for the whole space, and if $\{T_\alpha\}$ is a group of transformations of ω space into itself, taking measurable sets into measurable sets, $X(\alpha)=f(T_\alpha\omega)$ defines a random function on the group, that is, a family (parameter α) of random variables. If the transformations are measure preserving, the stochastic process so defined is stationary. [See also Kawada and Itô, Proc. Phys.-Math. Soc. Japan (3) **22**, 977–998 (1940); Kawada, ibid. (3) **23**, 669–686 (1941); these Rev. **2**, 223; **8**, 501.] *J. L. Doob.*

Horner, F. A problem on the summation of simple harmonic functions of the same amplitude and frequency but of random phase. Philos. Mag. (7) **37**, 145–162 (1946).

The probability density function $P_n(s)$ of $s=l\sum_{k=1}^n\cos\varphi_k$, where the φ_k's are independent random variables, uniformly distributed in $(0, 2\pi)$, is studied. For large n the distribution of s is approximately normal [this is an immediate consequence of the central limit theorem, although the author does not seem to be aware of this fact]. The author obtains $P_n(s)$ for $n=1$, 2, 3 and 7. For $n=1$ the result is trivial, for $n=2$ it can be expressed in terms of an elliptic integral. For $n=3$ a graphical procedure is adopted. For $P_7(s)$ a series expansion closely related to that of Pearson [Mathematical Contributions to the Theory of Evolution. XV. A Mathematical Theory of Random Migration, Draper's Co. Research Mem., Biometric Ser. 3, Cambridge University Press, 1906] is used. The problem arose in connection with the study of site errors in direction finding. *M. Kac* (Ithaca, N. Y.).

Fürth, R., and MacDonald, D. K. C. Statistical analysis of spontaneous electrical fluctuations. Proc. Phys. Soc. **59**, 388–408 (1947).

This paper is primarily devoted to the experimental verification of some of the basic distributions in the theory of random noise. A new theoretical contribution is an explicit calculation of the probability density $P(v; \tau)$ of the difference $v=R(t+\tau)-R(t)$, where $R(t)$ is the envelope of the noise record. The average $\Delta=\int_{-\infty}^{\infty}|v|P(v, \tau)dv$, which is analogous to Smoluchowski's "after effect," is also calculated and the result tested experimentally. *M. Kac.*

Batchelor, G. K. Kolmogoroff's theory of locally isotropic turbulence. Proc. Cambridge Philos. Soc. **43**, 533–559 (1947).

The author studies in detail the analysis of turbulence due to Kolmogoroff [C. R. (Doklady) Acad. Sci. URSS (N.S.) **30**, 301–305 (1941); **32**, 16–18 (1941); these Rev. **2**, 327; **3**, 221]; large eddies pass energy to small eddies, etc.; the smaller eddies dissipate more and more of their energies through the action of viscosity until the motion of the smallest becomes laminar. Kolmogoroff defines the concept of locally isotropic turbulence and is led to similarity hypotheses which make possible evaluation of the usual correlation functions. The author considers that some of Kolmogoroff's results are confirmed by experiment but many further results remain to be checked. *J. L. Doob.*

TOPOLOGY

✶**Fréchet, Maurice, et Fan, Ky. Introduction à la Topologie Combinatoire. I. Initiation.** Librairie Vuibert, Paris, 1946. viii+88 pp. 150 francs.

Entertaining reading about combinatorial topology accessible to a reader with very little mathematical preparation. The topics discussed are: the Jordan curve theorem, the map coloring problem, the Euler characteristic and the classification of surfaces. The proofs are very intuitive and are not intended to be complete. Many historical remarks are included. *S. Eilenberg* (New York, N. Y.).

Tutte, W. T. A family of cubical graphs. Proc. Cambridge Philos. Soc. **43**, 459–474 (1947).

The author defines an m-circuit of a graph to be a closed polygon formed by m distinct edges and m distinct vertices, and an s-arc to be a Hamiltonian path formed by s consecutive edges taken in a definite sense. The least m for which an m-circuit exists is called the girth of the graph. A connected graph is said to be s-regular if its group includes a unique operation transforming any s-arc into any other s-arc (e.g., reversing the sense of an s-arc). An ordinary regular graph is at least 1-regular, and the graph of edges and vertices of a cube is 2-regular. If an s-regular graph of degree three has E edges, its group is of order 2^sE, this being the number of s-arcs. The author proves that a graph of degree 3 cannot be s-regular for $s>5$, and that such a graph of girth m cannot be s-regular for $s>\frac{1}{2}m+1$. A graph of girth m with $s=[\frac{1}{2}m+1]$ is called an m-cage. He proves that, for such a graph, $E=3(3\cdot2^{s-2}-1)$ or $3(2^{s-1}-1)$ according as m is odd or even. He finds an m-cage for each $m<7$ and one for $m=8$, and proves that these are the only possibilities. The 2-cage $(s=2)$ merely consists of 3 (curved) edges all joining the same 2 vertices. The 3-cage $(s=2)$ is formed by the 6 edges and 4 vertices of a tetrahedron. The 4-cage $(s=3)$ is the Thomsen graph, whose 6 vertices, say 1, 2, 3, 4, 5, 6, are joined by 9 edges ij, where i is odd and j even. The 5-cage $(s=3)$ is the Petersen graph, whose 15 edges can be identified with the pairs of opposite edges of a regular dodecahedron; its group, of order $2^3\cdot15=120$, is symmetric. The 6-cage $(s=4)$ can be embedded into a torus

(in various ways) so that its 21 edges form Heawood's map of seven hexagons each surrounded by all the others; its group, of order $2^4 \cdot 21 = 336$, has a simple subgroup of order 168. Finally, the 8-cage ($s=5$) appears to be quite new. Since it has 45 edges, its group is of order 1440. This is actually the group of automorphisms of the symmetric group of degree 6. *H. S. M. Coxeter* (Toronto, Ont.).

Katětov, M. On topological spaces containing no disjoint dense sets. Rec. Math. [Mat. Sbornik] N.S. **21**(63), 3–12 (1947). (Russian. English summary)

Many results of this paper are contained in the reviewer's thesis [Duke Math. J. **10**, 309–333 (1943); these Rev. **5**, 46], which, as the author remarks, was not available to him until recently. A number of new results are proved. Typical new theorems are the following (the terminology is that introduced by the reviewer): (1) there exist maximal normal spaces of arbitrary infinite cardinal number (a question left unsolved by the reviewer); (2) any regularly maximal regular space is completely regular; (3) any bounded continuous real-valued function defined on a dense subset of a maximal Hausdorff space X can be continuously extended over X. The author raises this question: does there exist a Hausdorff space devoid of isolated points on which every real-valued function is continuous at some point? [Reviewer's note. It is obvious that any Hausdorff space with this property cannot contain complementary dense subsets; on the other hand, an example given by the author shows that Hausdorff spaces exist which have no complementary dense subsets but which do allow everywhere discontinuous mappings into the real number system.] *E. Hewitt* (Chicago, Ill.).

Marczewski, Edward. Séparabilité et multiplication cartésienne des espaces topologiques. Fund. Math. **34**, 127–143 (1947).

The author considers various properties of T_1-spaces which are all equivalent to Hausdorff's second axiom of countability for metric spaces, but which fail to be equivalent to this axiom for more general spaces. Let X be any T_1-space. The following properties of X are studied: (B) Hausdorff's second axiom of countability obtains in X; (D) X contains a countable dense subset; (K) the family Θ of open subsets of X is such that every uncountable subfamily of Θ contains an uncountable sub-subfamily consisting entirely of sets which have pairwise intersections nonvoid; (S) the family Θ of open subsets of X contains no uncountable subfamily of sets which are all pairwise disjoint. (Three other properties are also discussed.) It is observed that the implications (B)→(D)→(K)→(S) obtain for all T_1-spaces, and that, if X is metrizable, then (S)→(B) also obtains. Examples are adduced or cited to show the nonequivalence of (B), (D), (K) and (S), with the exception of (K) and (S). The author leaves as an open problem the existence or nonexistence of a T_1-space enjoying property (S) but failing to enjoy property (K).

The invariance or non-invariance of these properties under continuous mappings and under the formation of Cartesian products is discussed in some detail. One major theorem is the following. Let $\{X_\lambda\}$, $\lambda\epsilon\Lambda$, be any nonvoid family of nonvoid T_1-spaces. (The restriction to T_1-spaces is not essential.) Then the Cartesian product $\mathbf{P}X_\lambda$, $\lambda\epsilon\Lambda$, enjoys the property (K) if and only if each of the spaces X_λ enjoys the property (K). Applications of the invariance theorems are made to Cartesian products $\mathbf{P}N_\lambda$ and $\mathbf{P}B_\lambda$, where N_λ is a countably infinite discrete space (for all $\lambda\epsilon\Lambda$) and B_λ is a T_1-space consisting of precisely two points (for all $\lambda\epsilon\Lambda$). The author leaves open the question as to whether or not the Cartesian product of two spaces both enjoying property (S) also enjoys property (S).

The author remarks that properties (S) and (K) are meaningful for an arbitrary family of subsets of an arbitrary set, and proves that the subsets of the unit interval which are Lebesgue measurable and have positive measure are a family enjoying property (K). A construction due to Sierpiński is cited to show the nonequivalence of properties (S) and (K) for general families of sets [Ann. Scuola Norm. Super. Pisa (2) **2**, 285–287 (1933)]. *E. Hewitt.*

Wuytack, F. Les transformations bicontinues d'un espace topologique. Simon Stevin **25**, 142–145 (1947).

The characterization of homeomorphisms $T: A \leftrightarrow B$ among functions $T: A \to B$ in terms of commutation relations with the closure and complement operators is established. *R. Arens* (Los Angeles, Calif.).

Ramanathan, A. Maximal-Hausdorff spaces. Proc. Indian Acad. Sci., Sect. A. **26**, 31–42 (1947).

The author shows that the properties of (i) bicompactness and (ii) having no stronger Hausdorff topology, for a Hausdorff topology of a set, are not equivalent. Among other results, it is shown that (ii) implies semi-regularity. The present work seems to duplicate sections of a paper of M. Katětov [Časopis Pěst. Math. Fys. **69**, 36–49 (1940); these Rev. **1**, 317]. *R. Arens* (Los Angeles, Calif.).

Čech, Eduard, and Novák, Josef. On regular and combinatorial imbedding. Časopis Pěst. Mat. Fys. **72**, 7–16 (1947). (English. Czech summary)

The Wallman [Ann. of Math. (2) **39**, 112–126 (1938)] compactification ωQ of a topological space Q is shown to be characterized uniquely by the following two properties: (1) Q is regularly imbedded in ωQ, i.e. the family $\{\bar{F}\}$, where $\bar{F}$ is the closure in ωQ of a closed set F of Q, is a basis for the closed sets in ωQ; (2) Q is combinatorially imbedded in ωQ, i.e. if F_1 and F_2 are disjoint closed sets of Q then $\bar{F}_1 \cap \bar{F}_2 = 0$. Numerous examples are given to illustrate the definitions introduced. *S. Eilenberg* (New York, N. Y.).

Puckett, William T. A problem in connected finite closure algebras. Duke Math. J. **14**, 289–296 (1947).

Let $C(E)$ denote the "closure algebra" of all closed subsets of the unit interval $E: 0 \leqq x \leqq 1$, under Boolean operations and the closure operation $x \to \bar{x}$ [for terminology, see J. C. C. McKinsey and A. Tarski, Ann. of Math. (2) **45**, 141–191 (1944); these Rev. **5**, 211]. It is shown that any finite "connected" abstract "closure algebra" C is isomorphic with a subalgebra of $C(E)$. The proof is based on interior transformations, as defined by G. T. Whyburn ["Analytic Topology," Amer. Math. Soc. Colloquium Publ., v. 28, New York, 1942, p. 129; these Rev. **4**, 86]. It is shown that (i) there exists an interior transformation of E onto any finite connected T_0-space in which the void set is closed; (ii) if $f(S) = T$ is any interior transformation of S onto T, then the closure algebra $C(T)$ of T is isomorphic with a subalgebra of $C(S)$. More generally, if G is any locally connected separable metric space M, which is the union of all arcs joining some two of its points, then any finite connected closure algebra is isomorphic with a subalgebra of $C(M)$. *G. Birkhoff* (Cambridge, Mass.).

Bing, R. H. Skew sets. Amer. J. Math. 69, 493–498 (1947).

The author introduces skew sets of type 1 and type 2 which are patterned, respectively, after the two well-known nonplanar skew curves of Kuratowski. For a space S in which components of all open sets are arcwise connected it is shown that the presence of a skew set of type 1 (type 2) implies the presence of a skew curve of type 1 (either type 1 or type 2). It thus follows that the plane does not contain any skew set of either type. On this rests the proof that no plane set G contains a collection of five mutually separated point sets such that the closure of the sum of any pair of these five sets is the closure of a connected subset of G which is open in G. A construction is given of a skew set of type 2 lying within a skew curve of type 1.

W. W. S. Claytor (Hampton, Va.).

GEOMETRY

Schubarth, E. Der Gruppenbegriff in der Geometrie. Experientia 3, 385–393 (1947).

Freudenthal, H., and v. d. Waerden, B. L. On an assertion of Euclid. Simon Stevin 25, 115–121 (1947). (Dutch)

In a remark following Prop. 18 in Book XIII, Euclid asserted that "no other figure, besides the said five [Platonic] figures, can be constructed which is contained by equilateral and equiangular figures equal to one another." The assertion is true for all kinds of faces except triangles, but it is disproved for triangular faces by the dipyramid formed by placing two regular tetrahedra base to base. This is one of five such irregular convex polyhedra. The remaining four have 4 triangles at some vertices and 5 at others. If there are a_4 and a_5 vertices of these respective kinds, Euler's formula yields $2a_4+a_5=12$. The solution $a_4=1$, $a_5=10$ is easily seen to be impossible; but the values $a_4=2$, 3, 4, 5 are all significant. When $a_4=2$ we have a square antiprism with two square pyramids (halves of an octahedron) stuck onto its bases. The case $a_4=3$ can be derived from this by removing one of the pyramids and dividing the exposed square by a diagonal into two triangles which are afterwards distorted until they become equilateral. For $a_4=4$ the second pyramid is likewise removed, leaving the "Siamese dodecahedron" which has the symmetry of a tetragonal disphenoid. Finally, when $a_4=5$ we have the pentagonal dipyramid.

H. S. M. Coxeter.

Simpson, Harold. On the nodes of a rational plane curve. Math. Gaz. 31, 161–163 (1947).

It is well known that a rational plane curve $x:y:z=f_1(t):f_2(t):f_3(t)$, where the f_i are polynomials of degree n (or where one or two of them may be of lower degree) has the equivalent of $\frac{1}{2}(n-1)(n-2)$ double points. These are determined by solving the simultaneous equations $f_i(t)=f_i(s)$, $t\neq s$. The author presents a novel method of effecting this solution, using a determinant of order $n+1$, which appears to be preferable to the standard method which may be found in any treatise on higher plane curves.

R. A. Johnson (Brooklyn, N. Y.).

Claeys, A. On the skew strophoid. Simon Stevin 25, 122–141 (1947). (Dutch)

This is an exhaustive study of the rational cubic curve $(x^2+y^2)(gx+fy)=2axy$, which is the pedal of a parabola with respect to an arbitrary point on the directrix. The author gives several constructions, of which the following is simplest. Let a variable chord OA_1 of a given circle meet a fixed diameter in A. On this chord take a point P such that $OP=AA_1$. The locus of P is the curve under consideration. He also gives as many as ten different constructions for the tangent at P to this curve, and two for the center of curvature. He finds that the only real point of inflexion lies on the tangent to the given circle at the point diametrically opposite to O, though its position on that tangent involves a cubic irrationality.

H. S. M. Coxeter (Toronto, Ont.).

Thébault, Victor. Concerning the Euler line of a triangle. Amer. Math. Monthly 54, 447–453 (1947).

Thébault, Victor. La sphère de Longchamps d'un polyèdre dont les sommets sont cosphériques. C. R. Acad. Sci. Paris 225, 426–428 (1947).

Piggott, H. E., and Steiner, A. Isogonal conjugates. A new approach to certain geometrical theorems and to a general theory of conics. Math. Gaz. 31, 130–144 (1947).

This is an expository paper on the relations of isogonal conjugate points with reference to a triangle, treated by both synthetic and analytic methods. The latter treatment includes a brief excursus dealing with the quadratic transformation of which the relation is a special case. There is little that is new.

R. A. Johnson (Brooklyn, N. Y.).

Mesmer, Candida. Sui sistemi semplicemente infiniti di omografie piane che contengono un'omografia degenere. Boll. Un. Mat. Ital. (3) 2, 28–34 (1947).

Un système continu à un paramètre d'homographies planes peut contenir des homographies simplement, ou doublement, dégénérées. Les homographies du système, infiniment voisines d'une homographie singulière, définissent un repère canonique, et leurs premiers invariants dans le système s'interprètent géométriquement par des birapports de droites, ou de coniques.

P. Belgodère (Paris).

Algebraic Geometry

Zariski, Oscar. The concept of a simple point of an abstract algebraic variety. Trans. Amer. Math. Soc. 62, 1–52 (1947).

The classical definition of a simple point, by means of the rank of a Jacobian matrix, is not intrinsic (i.e., it seems to depend upon the given embedding of the variety in affine space). The author has [Amer. J. Math. 62, 187–221 (1940); these Rev. 1, 102] given an intrinsic definition of which he showed that it is equivalent to the classical one in the case of characteristic 0 (more generally, over a perfect groundfield), but weaker otherwise. He now takes up the subject again and clears it up completely. Let V be an irreducible variety (in the author's terminology, i.e. defined over a given ground-field k, by a prime ideal in a ring of polynomials over k); let $F(V)$ be the field of algebraic functions on V (with k as ground-field). Let W be an irreducible subvariety of V, $\mathfrak{o}=Q(W, V)$ the quotient-ring of W in $F(V)$ (i.e., the ring of elements of $F(V)$ which are finite at a generic point of W), $\mathfrak{m}=m(W, V)$ the maximal ideal in $\mathfrak{o}$ (the ideal of nonunits in $\mathfrak{o}$, i.e. of elements of $F(V)$ which vanish on W); $\mathfrak{o}/\mathfrak{m}$ may be identified with the field $F(W)$ of algebraic functions on W, and the additive group $\mathfrak{m}/\mathfrak{m}^2$ may be regarded as a vector-space $M(W, V)$ over $\mathfrak{o}/\mathfrak{m}=F(W)$;

let τ be the canonical homomorphism of $\mathfrak{m}$ onto $\mathfrak{m}/\mathfrak{m}^2 = M(W, V)$; τ may be described as a "linearization in the neighborhood of W" and (taking into account the fact that it is defined only on $\mathfrak{m}$ and not on $\mathfrak{o}$) has the essential properties of a differentiation operator. The dual vector-space $T(W, V)$ to $M(W, V)$ over $F(W)$ seems to deserve to be called the tangent vector-space to V along W; if V' is a subvariety of V, containing W, $T(W, V')$ may be regarded as (canonically) embedded in $T(W, V)$. The dimension of $M(W, V)$ and of $T(W, V)$, over $F(W)$, is at least equal to dim $(V)-$dim (W); if it has exactly that value, W is called simple on V; if, then, elements u_i of $\mathfrak{m}$ are such that the τu_i are a basis for $M(W, V)$ over $F(W)$, they are called uniformizing parameters for V across W.

The first part of the paper clears up the geometric content of the definitions; in affine space S^n (or on an ambient variety of dimension n on which P is simple), a point P is simple on a variety V of dimension r if and only if V is, locally at P, a regular intersection of $n-r$ hypersurfaces (varieties of dimension $n-1$); the intersection is called regular if the tangent linear spaces to the hypersurfaces at P (considered as embedded in the tangent linear space at P to the given S^n or to the ambient variety) have an intersection of dimension r; furthermore, a subvariety W of a variety V is singular (i.e., nonsimple) if and only if all its points (over the ground-field k, i.e. defined by zero-dimensional prime ideals in the polynomial ring over k) are singular on V. Furthermore, W is simple on V if and only if the ring $\mathfrak{o}=Q(W, V)$ is "regular" (in the sense of Krull and Chevalley) and $\mathfrak{o}$ is then a unique factorization domain.

The second part of the paper gives criteria for simplicity in terms of Jacobian matrices. When $F(W)$ is separably generated over k, the operator τ is substantially identical with the ordinary differential, defined by means of partial derivations with respect to a suitable set of independent variables; from this it follows that the above definition of simplicity is then equivalent to the classical criterion in terms of the rank of the Jacobian matrix. When no such assumption is made on $F(W)$, then "nonseparable" derivations in $F(W)$ (arising from the derivations in k over k^p, where p is the characteristic) have to be taken into account; it is shown that, when this is properly done, the above results remain true; therefore, a criterion for simplicity can in all cases be given by means of the rank of a suitably defined Jacobian matrix, involving the "nonseparable" derivations as well as the usual ones. Among other consequences, it follows from this that the singular points of a variety V (for a given ground-field) are the points of a proper subvariety of V, the "singular locus" (which of course may be reducible). Finally, a point or subvariety of V is defined to be "absolutely simple" if it is simple and remains so by every extension of the ground-field; this is shown to be equivalent to the classical definition, and also to the following: W is absolutely simple on V if and only if V is, locally at W, analytically equivalent to a linear space of the same dimension; analytical equivalence is of course to be understood here in the sense of formal power-series, or rather of the completion of the local rings involved. *A. Weil.*

Hodge, W. V. D. Note on the conditions for a p-cycle of an algebraic manifold to be of rank k. Proc. Cambridge Philos. Soc. **43**, 577–580 (1947).

On an algebraic variety V^m without multiple point (over the complex field) a cycle Z, in the homology group of V^m of dimension p (with rational numbers as coefficient group), is called of rank k if it is homologous to a cycle lying on an algebraic subvariety U^{p-k} of V^m, of (complex) dimension $p-k$; the rank is always nonnegative; an outstanding problem is to find necessary and sufficient conditions for the rank to have a given value. The author [Theory and Applications of Harmonic Integrals, Cambridge University Press, 1941; these Rev. **2**, 296] has given necessary conditions, in terms of the periods of certain integrals. He now transforms these conditions so that they have the following form: if a cycle Z is of rank k, there exists a form of a certain type on V^m, homologous to Z in the sense of de Rham.

A. Weil (Chicago, Ill.).

Nagell, Trygve. Les points exceptionnels sur les cubiques planes du premier genre. Nova Acta Soc. Sci. Upsaliensis (4) **14**, no. 1, 34 pp. (1946).

Nagell, Trygve. Les points exceptionnels sur les cubiques planes du premier genre. II. Nova Acta Soc. Sci. Upsaliensis (4) **14**, no. 3, 40 pp. (1947).

Let C be a plane cubic of genus one. Let P_0 be a point of C. The tangent to C at P_0 cuts C again at a point P_1, the tangential of P_0. Let P_2 be the tangential of $P_1, \cdots, P_{r+1}$ the tangential of P_r. The point P_0 is said to be exceptional if the sequence $(P_0, \cdots, P_r, \cdots)$ has only a finite number of distinct terms. The object of these papers is to study the exceptional points of C with coordinates in some given field Ω of characteristic zero which contains the coefficients of some equation of C. The points of C whose coordinates are in Ω will be called rational. If C has at least one rational point, it is birationally equivalent over Ω to a Weierstrass cubic W of equation $y^2=4x^3-g_2x-g_3$ and we can parametrize C by means of the elliptic argument on W. Let $\{\omega, \omega'\}$ be a basic system of periods for the elliptic functions belonging to W. The condition for three points of elliptic arguments u_1, u_2, u_3 on C to be collinear is $u_1+u_2+u_3\equiv 3\theta$ (mod ω, ω'), where θ is a constant. The exceptional points of C are then the points $u+\theta$, where u is the argument of an exceptional point of W, and the arguments u of the exceptional points of W are the numbers such that $nu\equiv 0$ (mod ω, ω') for some integral $n\neq 0$. This shows that, if C has at least one rational point, it has exactly as many rational exceptional points as W, and that the rational exceptional points of W form a group. In case Ω is a field of finite degree over the rational numbers, the group of rational exceptional points of W is finite. The author indicates a method of determining this group completely when an upper bound for its order is known.

When Ω is the field Q of rational numbers, it is known that the group of rational exceptional points is either cyclic or product of two cyclic groups. If $\Omega=Q(\sqrt{\Delta})$, Δ an integer, then the author determines entirely the group of rational exceptional points for a cubic W which is of one of the forms $y^2=x^3-Ax$ or $y^2=x^3-B$; the result is too complicated to be stated here.

Let C_1 be the cubic $y^2=x^3-Ax-B$, where A and B are in Ω, and let C_2 be the cubic $y^2=x^3-A\Delta^2x-B\Delta^3$, where $\Delta\epsilon\Omega$, $\sqrt{\Delta}\bar{\epsilon}\Omega$. Assume that the groups G_1, G_2 of exceptional points in Ω of C_1, C_2 are of finite orders n_1, n_2; then the group of exceptional points of C_1 (or C_2) in $\Omega(\sqrt{\Delta})$ has for order one of the numbers n_1n_2, $\frac{1}{2}n_1n_2$ or $\frac{1}{4}n_1n_2$. The orders $\frac{1}{2}n_1n_2$, $\frac{1}{4}n_1n_2$ can occur only if n_1 is even; $\frac{1}{4}n_1n_2$ can occur only if all half periods belong to G_1. If C is a cubic which has no rational exceptional point in Ω, and if α is algebraic of degree not congruent to 0 (mod 3) over Ω, then C has no rational exceptional point over $\Omega(\alpha)$.

The author also gives the form which the invariants of a cubic must have if it is known that its exceptional group over a field Ω is of order divisible by one of the integers 3, 4, 5, 6 or 7. *C. Chevalley* (Princeton, N. J.).

Jongmans, F. Les limitations du nombre des modules des surfaces algébriques. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **31** (1945), 639–646 (1946).

The author obtains bounds, in terms of the invariants $p_a, p_g, p^{(1)}$, between which the number of moduli of an algebraic surface, not possessing a continuous group of birational transformations into itself, must lie. These bounds are an improvement on those formerly obtained by Segre [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (6) **19**, 488–494 (1934)]. *J. A. Todd* (Cambridge, England).

Pompilj, Giuseppe. Sui piani tripli con quartica di diramazione. Ann. Mat. Pura Appl. (4) **24**, 65–117 (1945).

Any triple plane with a quartic branch curve D^4 is known to be the projection, from a line g not meeting the surface, of a rational normal cubic scroll F^3 (or a rational cubic cone) of S_4. The present memoir is entirely devoted to a detailed examination of triple planes of this very special type, the object being to provide an experimental basis for the theory of triple planes in general. When F^3 is not a cone, D^4 is in general irreducible and has three cusps at the projections of the intersections of F^3 with the harmonic polar plane of g; but if g lies in a solid containing two consecutive generators of F^3, then D^4 breaks up into a cuspidal cubic and its inflexional tangent. If F^3 is a cone, D^4 consists of 4 concurrent lines, of which two may coincide. The author investigates the existence and uniqueness of the triple planes for an assigned branch curve of each of the above types; and he shows also how to construct the continuous systems of curves in the plane which represent linear systems of curves on the projective model. *J. G. Semple* (London).

Kasner, Edward, and De Cicco, John. General polar theory. Scripta Math. **13**, 53–57 (1947).

Expository article. *J. A. Todd* (Cambridge, England).

Derwidué, L. Sur les variétés algébriques à surfaces-sections rationnelles. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **33**, 251–261 (1947).

d'Orgeval, Bernard. Remarques sur les nappes réelles de certaines surfaces. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **33**, 248–250 (1947).

Differential Geometry

***Sauer, Robert. Finite Analoga zur Differentialgeometrie der Asymptotennetze.** Ber. Math.-Tagung Tübingen 1946, pp. 127–129 (1947).

A skew lattice $\mathfrak{L}$ is a one-to-one image in space of the sides and vertices of the plane square lattice. Each side of $\mathfrak{L}$ is a segment of a straight line. Thus, the image of each fundamental square is a skew quadrangle. It is assumed that $\mathfrak{L}$ is "ebeneckig," i.e., the four sides of $\mathfrak{L}$ that join at each vertex are coplanar. So, the end points of a side a of length $s(a)$ determine two planes. If $\varphi(a)$ is the angle between them, the torsion of a is defined to be $\pm\sin \varphi(a)/s(a)$. Similarly, a spherical image and a Gauss curvature of $\mathfrak{L}$ can be defined. The author states or indicates finite analogues of various theorems of differential geometry, such as Beltrami's and Enneper's formulas on the torsion of the asymptotic lines. A suitable limiting process transforms $\mathfrak{L}$ into the net of asymptotic lines of a surface of negative curvature. Few details and no proofs are given.

P. Scherk (Saskatoon, Sask.).

Marcus, F. Sopra una classe di reti e di superficie, in relazione con le congruenze di rette di Waelsh. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **2**, 408–410 (1947).

A congruence of straight lines is a Waelsh congruence if to the asymptotic lines of one focal surface there correspond a system of conjugate lines on the other focal surface. It is proved in this note that the conjugate tangents of a surface describe Waelsh congruences if, by means of a normalization, the equations of the surface in projective space can be reduced to a certain form. When the Darboux invariants of this special form are equal, the special net so obtained is called a Waelsh net. *E. T. Davies* (Southampton).

Haimovici, Adolf. Sur une classe de surfaces en relation avec les développables d'une congruence de droites et sur les surfaces ayant un réseau orthogonal de courbes planes. C. R. Acad. Sci. Paris **225**, 275–277 (1947).

Dans cette note l'auteur recherche les surfaces admettant un réseau orthogonal constitué par des courbes planes. Il démontre que si l'on se donne arbitrairement deux surfaces développables, il existe une surface et une seule passant par une courbe arbitraire de l'espace et sur laquelle les plans tangents aux deux développables déterminent un réseau du type considéré. Sous une autre forme: si l'on se donne arbitrairement une congruence de droites à focales développables et une courbe quelconque de l'espace, il existe une surface et une seule passant par la courbe et sur laquelle les développables de la congruence déterminent un réseau orthogonal. Il signale la généralisation immédiate suivante de la proposition qui précède: étant donnée une congruence quelconque de droites, par chaque courbe de l'espace il passe une surface sur laquelle les développables de la congruence déterminent un réseau orthogonal. La note se termine par l'indication de quelques cas particuliers, obtenus en demandant que les lignes du réseau soient de courbure, que la congruence qui intervient dans la construction soit normale ou à nappes focales curvilignes, ou que les courbes du réseau soient dans des plans passant par un point fixe.

P. Vincensini (Besançon).

Chern, Shiing-Shen. Some new viewpoints in differential geometry in the large. Bull. Amer. Math. Soc. **52**, 1–30 (1946). [MF 15215]

Dieser Vortrag [gehalten im September 1945] zeigt, dass wir uns am Beginn einer neuen Epoche in der "Differentialgeometrie im Grossen" befinden. Das Neue ist charakterisiert durch die Synthese der topologischen Theorie der Faser-Räume und der Cartan'schen Methode der Differentialformen in der Geometrie.

Nach einer kurzen programmatischen Einleitung [§ 1] berichtet der Verfasser knapp und klar über die Grassmann'sche Algebra [§ 2], Cartan's Differential-Methode [§ 3], Faserräume [§ 4], Riemann'sche Geometrie [§ 5]. In diesem § 5 wird Cartan's Methode in den Vordergrund gestellt; dabei wird über der n-dimensionalen Riemann'schen Mannigfaltigkeit M der Faserraum $\mathfrak{F}$ aller orthogonalen n-Beine von Einheitsvektoren betrachtet; die bekannten Formen ω_i, ω_{ij} sind als Formen in $\mathfrak{F}$ aufzufassen.

Im nächsten Abschnitt [§ 6] wird an dem "Formenproblem" (d.h. der Frage, wann zwei definite Formen $g_{ij}dx^i dx^j$ und $g_{ij}^* dx^{*i} dx^{*j}$ isometrische Riemann'sche Geometrien definieren) gezeigt, wie natürlich und notwendig die Einführung des Faserraumes $\mathfrak{F}$ (dessen Faser hier als orthogonaler Gruppenraum gedeutet wird) ist, sogar für die Differentialgeometrie "im Kleinen." Der analoge Prozess ist nicht nur der Riemann'schen Geometrie, sondern auch anderen geometrischen Theorien angemessen; dies wird durch das Beispiel der Wege-Geometrie (geometry of paths) illustriert.

Der Abschnitt 7 handelt von den Beziehungen zwischen dem Faserraum $\mathfrak{F}$, den mit $\mathfrak{F}$ "assoziierten" Faserräumen $\mathfrak{F}^{(p)}$, welche aus den Systemen von je p orthogonalen Einheitsvektoren über M bestehen, und der Basis-Mannigfaltigkeit $M=\mathfrak{F}^{(0)}$. Insbesondere werden (unter Benutzung des Theorems von de Rham, nach welchem man die Cozyklen mit den Differentialformen identifizieren darf) die Beziehungen diskutiert, welche durch die natürliche Projektion von $\mathfrak{F}$ auf einen Raum $\mathfrak{F}^{(p)}$ zwischen den Differentialformen dieser Räume vermittelt werden. Eine solche Betrachtung ist unter anderem wesentlich für das Verständnis der Allendoerfer-Weil-Chern'schen Verallgemeinerung der Gauss-Bonnet'schen Formel.

Der § 8 behandelt einen Satz aus der "Differentialgeometrie im Grossen" im eigentlichen Sinne, nämlich den folgenden Satz von Pontrjagin: Für eine geschlossene M^n im euklidischen Raum E^{n+N} (wobei die Metrik der M^n durch den E^{n+N} bewirkt ist) lassen sich die (ganzzahligen) charakteristischen Cohomologieklassen (im Sinne von Stiefel-Whitney) durch Integrale der sogenannten "Krümmungsformen" der M^n ausdrücken. Dabei werden die Abbildungen der M^n in den Grassmann'schen Raum der n-dimensionalen Ebenen durch einen Punkt des E^{n+N} benutzt, welche die sphärische Abbildung von Gauss verallgemeinern; es wird auf den Zusammenhang dieser Abbildungen mit dem Satz von Whitney-Steenrod hingewiesen, welche die Sphärenräume über einer beliebigen M mit den Abbildungen der M in Grassmann'sche Mannigfaltigkeiten in Verbindung bringt.

Nimmt man Hermite'sche statt beliebiger Riemann'scher Mannigfaltigkeiten [§ 9], so wird aus verschiedenen Gründen die Situation einfacher als diejenige, die im § 8 vorliegt; insbesondere gilt das Analogon des Satzes von Pontrjagin dann auch ohne die Voraussetzung, dass die Metrik der M durch einen euklidischen Raum, in welchem M liegt, bewirkt wird.

Schliesslich [§ 10] wird kurz darauf hingewiesen, dass die Methoden, von denen die Rede war, für die Untersuchung des Zusammenhanges der Differentialgeometrie nicht nur mit topologischen, sondern auch mit anderen geometrischen Eigenschaften "im Grossen" wertvoll sind. Dies wird durch das Beispiel der "kinematischen Dichte" und der sogenannten kinematischen Hauptformel in der Integralgeometrie illustriert, woraus man sieht, dass selbst die klassische isoperimetrische Ungleichung in der Ebene etwas mit Differentialformen und Faserräumen zu tun hat.

H. Hopf (Zürich).

Lichnerowicz, André. Sur les formes harmoniques de certains espaces fibrés. C. R. Acad. Sci. Paris **224**, 1413–1414 (1947).

The author considers two compact orientable Riemannian manifolds V and E of dimensions n and $n+q$; E is a fiber bundle over V, with fiber map φ; the metric of V is given, in Cartan's notation, by $ds^2=\sum_1^n(\omega_i)^2$; the metric of E is assumed to be of the form $d\sigma^2=\sum_1^n(\omega_i)^2+\sum_1^q(\pi_\alpha)^2$. For any differential form α (in V or E), $*\alpha$ is the harmonic conjugate in the sense of Hodge and de Rham; $d\alpha$ denotes the exterior derivative of α; $\varphi^{-1}\alpha$ is the inverse image under φ. Define $d^*=(-1)^{m(p+1)+1}*d*$, where $m=\dim V$ or E, $p=\deg\alpha$. The relation $d\varphi^{-1}-\varphi^{-1}d=0$ holds. The author shows that, putting $\theta=\pi_1\cdots\pi_q$, one gets $(d^*\varphi^{-1}-\varphi^{-1}d^*)\alpha=\pm*(d\theta\cdot\varphi^{-1}*\alpha)$. For 1-forms this is shown to be 0; for the Betti numbers R_p and ρ_p of V and E it follows that $R_1\leqq\rho_1$. If θ is harmonic then d^* and φ^{-1} commute, the system $\pi_\alpha=0$ is completely integrable, the φ^{-1}-image of every harmonic form is harmonic, and one has $R_p\leqq\rho_p$.

H. Samelson.

Ruse, H. S. On simply harmonic spaces. J. London Math. Soc. **21** (1946), 243–247 (1947).

Lichnerowicz avait émis l'hypothèse [Bull. Soc. Math. France **72**, 146–168 (1944); ces Rev. **7**, 80] que, pour $n=4$, les espaces riemanniens harmoniques V_n étaient symétriques au sens d'É. Cartan. Depuis lors, A. G. Walker [J. London Math. Soc. **21**, 47–57 (1946); ces Rev. **8**, 351] a construit une théorie des espaces harmoniques V_n qui sont en même temps symétriques et a fourni des exemples d'espaces V_4 dont la courbure n'est pas constante mais qui sont effectivement symétriques. Ruse apporte ici un contre-exemple donné par la métrique $ds^2=2dxdz+2ydxdt+2xdydt+F(y)tdy^2$, où $F(y)$ est une fonction arbitraire. Cet espace est simplement harmonique, selon la terminologie de Walker, en ce sens que l'équation de Laplace associée admet la solution élémentaire s^{-2}. Il satisfait d'autre part à la relation

$$\nabla_p R_{ijkl}=k_p R_{ijkl},\qquad k_p=\partial_p\log|F(y)|, \tag{1}$$

et par suite n'est pas symétrique en général. La méthode utilisée est celle de Walker [Proc. Edinburgh Math. Soc. **7**, 16–26 (1942); ces Rev. **4**, 171]. Ruse attire l'attention sur l'intérêt des espaces K_n dont le tenseur de Riemann-Christoffel satisfait à une relation du type (1). On notera que l'exemple ci-dessus admet la signature $++--$, alors que, dans son travail cité, Lichnerowicz n'envisageait que des espaces à métrique définie positive. En ce qui concerne ces derniers espaces, la question reste ouverte.

A. Lichnerowicz (Strasbourg).

Hombu, Hitoshi, and Mikami, Misao. Conics in the projectively connected manifolds. Mem. Fac. Sci. Kyūsyū Imp. Univ. A. **2**, 217–239 (1942).

Hombu et Okada ont récemment étendu [Proc. Phys.-Math. Soc. Japan (3) **23**, 357–362 (1941); ces Rev. **3**, 20] la théorie des changements projectifs de connexion due à T. Y. Thomas [Math. Z. **25**, 723–733 (1926)] à des connexions affines non symétriques (Γ^i_{jk}) d'une variété à n dimensions V_n. La méthode de Thomas conduit à associer à la variété donnée une variété V_{n+1} douée de la connexion affine

$$\begin{cases} {}^*\Gamma^i_{jk}=\Pi^i_{jk};\quad {}^*\Gamma^0_{jk}=\dfrac{n+1}{n-1}B_{jk}, & i, j, \text{etc.}=1,\cdots,n;\\[2ex] {}^*\Gamma^\lambda_{0\mu}={}^*\Gamma^\lambda_{\mu 0}=-\dfrac{1}{n+1}\delta^\lambda_\mu, & \lambda,\mu,\text{etc.}=0,1,\cdots,n,\end{cases}$$

où les Π^i_{jk} sont les paramètres projectifs de Veblen et Thomas et où B_{jk} est le tenseur contracté de courbure projective. Les équations différentielles $({}^*\delta^2x^\lambda/dt^2=0)$, où ${}^*\delta$ représente la différentiation covariante selon la connexion $({}^*\Gamma^\lambda_{\mu\nu})$, déterminent dans V_n un système de courbes qui est dit par les auteurs le système des coniques relatifs à la connexion. Par un choix convenable de la variable auxiliare x^0, on peut s'assurer que le paramètre t se trouve défini à une transfor-

mation homographique près. Les auteurs étudient le contact des coniques attachées à deux connexions différentes, ainsi que le contact d'une conique avec un parabole attachée à (Γ^i_{jk}) $(\delta^3 x'^i/dt^3=0)$. Ainsi, étant donnée une parabole arbitraire relative à (Γ^i_{jk}), il existe une conique et une seule de la connexion $({}^*\Gamma^\lambda_{\mu\nu})$ associée à (Γ^i_{jk}) telle que la conique ait un contact du quatrième ordre avec la parabole donnée. On en déduit que pour que toutes les paraboles d'une connexion affine symétrique (Γ^i_{jk}) soient des coniques de la connexion subordonnée $({}^*\Gamma^\lambda_{\mu\nu})$, il faut et il suffit que le tenseur contracté de courbure de (Γ^i_{jk}) soit nul.

A. Lichnerowicz (Strasbourg).

Rosenfeld, B. Theory of surfaces in symmetrical spaces. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **9**, 371–386 (1945). (Russian. English summary)

In earlier papers the author had considered the representations of m-parameter varieties consisting of configurations of a Euclidean space (for instance, the totality of spheres) as an m-dimensional surface in a space of sufficiently high dimension. Here a general theory is given in which a variety M with a transitive Lie group $\mathfrak{G}$ is imbedded into a Euclidean or pseudo-Euclidean space E. If every transformation of $\mathfrak{G}$ may be continued into a rotation of E the author says that we have a "one-sided" representation of M as a "surface" of revolution in E; if, in addition, every rotation of E which takes M into itself induces in M a transformation of the group $\mathfrak{G}$ he speaks of a "two-sided" representation. The greater part of the paper is devoted to the case when M is symmetric in the sense of É. Cartan, and is based on a series of Cartan's papers.

In general, if the group is compact the imbedding space is Euclidean and if the group is noncompact (in this case it is necessary to assume that it is linearly representable) E is pseudo-Euclidean (has an indefinite metric). We have a two-sided representation when the group is semisimple, and if that is not the case the representation is one-sided. The method, in general, consists in introducing into the algebra representing the group $\mathfrak{G}$ of a Euclidean or pseudo-Euclidean metric. The author considers, in particular, the representation of the orthogonal group in the algebra of real matrices, the unimodular unitary group in the algebra of complex matrices and the spinor group in the algebra of Clifford numbers. The geometrical examples include the set of m-planes through a point of Euclidean or pseudo-Euclidean n-space, m-dimensional spheres of n-space, hyperquadrics of projective n-space. In the last section homogeneous nonsymmetric spaces are briefly considered, such as the variety consisting of n-simplices of projective $(n-1)$-space and the variety consisting of sets of m mutually perpendicular unit vectors in Euclidean or pseudo-Euclidean n-space.

G. Y. Rainich (Ann Arbor, Mich.).

NUMERICAL AND GRAPHICAL METHODS

Comrie, L. J. Calculating—past, present and future. Future, Overseas Number, 61–69 (1947).

Hartree, D. R. Recent developments in calculating machines. J. Sci. Instruments **24**, 172–176 (1947).

***Murray, Francis J. The Theory of Mathematical Machines.** King's Crown Press, New York, 1947. viii+116 pp. $3.00.

The purpose of this work is the presentation and discussion of the mathematical aspects of computing machinery. The treatment is largely with respect to component parts such as counters, adders, integrators, etc. One section is devoted to composite machines. For much the most part, the devices are of continuous or analogue type, that is, nondigital. The work is in four parts: (I) digital machines, component parts of desk calculators, etc.; (II) continuous operators, component analogue devices of all types; (III) the solution of problems, composite analogue devices such as differential analyzers, network analyzers, linear equation solvers, etc.; (IV) mathematical instruments, planimeters, harmonic analyzers, etc. The book contains nothing on large scale discrete variable calculators. There are about 200 original line drawings.

D. H. Lehmer (Berkeley, Calif.).

***Tables of Spherical Bessel Functions.** Prepared by the Mathematical Tables Project, National Bureau of Standards. Vol. II. Columbia University Press, New York, 1947. xx+328 pp. $7.50.

[For vol. I cf. these Rev. **8**, 406.] The introduction contains sections on the functions $(\frac{1}{2}\pi/x)^{\frac{1}{2}}J_\nu(x)$ and $\Lambda_\nu(x)=\Gamma(\nu+1)(\frac{1}{2}x)^{-\nu}J_\nu(x)$, on the zeros of $J_\nu(x)$ and $J_\nu'(x)$ and on the interpolation of the tables. The tabular material of the present volume is as follows.

Tables of $(\frac{1}{2}\pi/x)^{\frac{1}{2}}J_\nu(x)$ for $\pm\nu=14.5(1)21.5$, $x=0(.01)10(.1)25$; and for $\pm\nu=22.5(1)30.5$, $x=10(.1)25$. The entries are given to 8 to 10 significant figures, except for a few entries near the zeros of the functions. Second differences (sometimes modified) and in some places fourth differences are also given.

Tables of $\Lambda_\nu(x)$ for $\nu=0.5(.5)20.5(1)30.5$, $x=0(.1)10$, 9D; for $\nu=0.5(1)30.5$, $x=10(.1)25$, mostly 7D; for $-\nu=14.5(1)16.5$, $x=0(.1)9.5(.05)10(.1)25$, mostly 7S; and $-\nu=17.5(1)30.5$, $x=0(.1)25$, mostly 7S; with (sometimes modified) second differences and in places with (natural or modified) fourth differences.

Zeros of $J_\nu(x)$ with corresponding values of $J_\nu'(x)$ for $\nu=-21.5(1)19.5$; and zeros of $J_\nu'(x)$ with corresponding values of $J_\nu(x)$ for $\nu=-19.5(1)22.5$; both to 5 to 9 decimals. With increasing $|\nu|$ the number of zeros given decreases from 8 to 1.

Tables of interpolation coefficients for Everett's formula.

A. Erdélyi (Pasadena, Calif.).

***Table of the Bessel Functions $J_0(z)$ and $J_1(z)$ for Complex Arguments.** Prepared by the Mathematical Tables Project, National Bureau of Standards. 2d ed. Columbia University Press, New York, 1947. xliv+403 pp. $7.50. have 1943

Since the publication of the first edition of this volume in 1943 [these Rev. **5**, 159] no errors have been reported in the tabular material. However, . . . certain errors in the labeling of the graphs on pages xv and xvii . . . have been corrected in the present edition. Some minor revisions have been made in the introduction; in particular, the revised introduction includes explicitly the relations between the tabulated functions on the 45° ray and the ber and bei functions.

From the preface.

Neĭšuler, L. Ya. On the tabulation of functions given in implicit form. Bull. Acad. Sci. URSS. Cl. Sci. Tech. [Izvestia Akad. Nauk SSSR] **1947**, 597–608 (1947). (Russian)

The author continues his study of the problems involved in the tabulation of systems of nonexplicit functions of two or more variables. The case considered in a previous paper [C. R. (Doklady) Acad. Sci. URSS (N.S.) **44**, 360–364 (1944); these Rev. **7**, 83] is now illustrated. Part two of the paper considers the problem of tabulating the function $v=f(x, y, z, u)$ defined implicitly by the functional relation $\varphi(\eta(v, x), y)=\psi(\mu(v, z), u))$, where the ratio η/μ or the difference $\eta-\mu$ is independent of v. The problem is solved by constructing a number of auxiliary tables of single and double entry. Applications to punch card techniques are mentioned. [See also C. R. (Doklady) Acad. Sci. URSS (N.S.) **36**, 121–124 (1942); **43**, 142–146 (1944); these Rev. **4**, 202; **6**, 132.] *D. H. Lehmer* (Berkeley, Calif.).

Valiron, G. Note sur l'interpolation. Mémorial de l'Artillerie Française **20**, 149–161 (1946).

The author derives several formulas for interpolation by spliced polynomial arcs, all special cases of Hermite's interpolation formula, and including the classical Karup-King and Sprague formulas, but without any indication that the formulas derived were previously known. He then considers the case of interpolation for functions of two variables and obtains a formula for interpolating for arguments (x, y) within the square $0\leqq x\leqq 1$, $0\leqq y\leqq 1$, given the values of the function and its first partial derivatives at the four vertices of the square, using polynomials of the third degree in x and y separately, such that the function and its first partial derivatives are continuous when the formula is applied to adjacent squares. A similar formula is derived which requires as given data values of certain second partial derivatives, uses fifth degree polynomials, and provides for continuity of both first and second partial derivatives.
T. N. E. Greville (Washington, D. C.).

Nielsen, K. L., and Goldstein, L. An algorithm for least squares. J. Math. Phys. Mass. Inst. Tech. **26**, 120–132 (1947).

The authors give a method and explicit formulas, which are slightly different from the usual ones, for fitting a series of points of an empirical curve with equidistant abscissas. An extensive table gives some auxiliary quantities for the approximating parabolas of the degrees 2, 3, 4 if the number of the given points is less than or equal to 100.
E. Bodewig (The Hague).

Čupr, Karel. On logistic growth. Věstník Královské České Společnosti Nauk. Třída Matemat.-Přírodověd. **1946**, no. 7, 16 pp. (1947). (Czech)

The "logistic curves" are solutions of $x'=b(x-a)(c-x)$. The cross ratio of four x-values corresponding to equidistant t-values is constant. It is shown how this property can be used in fitting curves to empirical data. *W. Feller.*

✶Kamela, Czesław. Einschneiden mit der Doppelrechenmaschine. Sammlung wissenschaftlicher Arbeiten der in der Schweiz internierten Polen, Band 1, Heft 3, pp. 50–77. Eidg. Kommissariat für Internierung und Hospitalisierung, 1943.

In der Vermessungspraxis spielt die Koordinatenbestimmung durch Einschneiden eine sehr wichtige Rolle. Zur Lösung auf einer Doppelrechenmaschine eignet sich am besten die graphisch-analytische Methode. Zuerst werden die Maschinentypen, nachher die für die Maschinenberechnung am besten geeigneten Methoden besprochen.
Extract from the paper.

Ergen, William Krasny. Bridge type electrical computers. Rev. Sci. Instruments **18**, 564–567 (1947).

If three legs of a Wheatstone bridge have the resistances x_1, x_2 and x_3, then by balancing the bridge the fourth leg may be adjusted to give x_1x_2/x_3. Replacing each resistor by several resistors in series to represent a linear transformation of several variables, the bridge may be used to solve quadratic equations. It is shown that such bridge arrangements correspond only to quadratic equations with rank 4 and signature 0. By employing combinations of parallel and series resistors in the legs, more general quadratic and higher order equations may be solved. *G. Kron.*

Fairthorne, R. A. Mechanical instruments for solving linear simultaneous equations. Ministry of Supply [London], Aeronaut. Res. Council, Rep. and Memoranda no. **2144** (**8311**), 7 pp. (1944).

Some mechanical instruments are reviewed, the principles used ranging from nomography to hydrostatics. It is concluded that, while almost any mechanical principle can be employed, the instrument can be successful only if carried out as a major engineering project.
From the author's summary.

✶Quade, W. Auflösung linearer Gleichungen durch Matrizeniteration. Ber. Math.-Tagung Tübingen 1946, pp. 123–124 (1947).

Let the system of equations $\mathfrak{A}\mathfrak{x}=\mathfrak{b}$ be normalised: $\mathfrak{S}\mathfrak{x}=\mathfrak{A}'\mathfrak{b}$, where $\mathfrak{S}=\mathfrak{A}'\mathfrak{A}$. Form the vectors $\mathfrak{x}_m$ by means of $\mathfrak{x}_{m+1}=\mathfrak{C}\mathfrak{x}_m+\mathfrak{c}$, where $\mathfrak{x}_0$ is arbitrary, $\mathfrak{C}=E-2\alpha\mathfrak{S}$, α real, $0<\alpha<S^{-1}$, $S=|\sum s_{ik}^2|^{\frac{1}{2}}$, $\mathfrak{c}=2\alpha\mathfrak{A}'\mathfrak{b}$. Then $\lim \mathfrak{x}_m=\mathfrak{y}$ will exist and will yield a solution of $\mathfrak{S}\mathfrak{x}=\mathfrak{A}'\mathfrak{b}$ if such a solution exists.
E. Bodewig (The Hague).

Kochmański, Tadeusz. The Cracovians as a new branch of applied mathematics. Cracow Observatory Reprints, no. **23**, 3 pp. (1947). (Polish. English summary)
Reprinted from Życie Nauki **3**, no. 17–18.

Biezeno, C. B., and Bottema, O. The convergence of a specialized iterative process in use in structural analysis. Nederl. Akad. Wetensch., Proc. **49**, 489–499 (1946).

Byhovskiĭ, M. L. The cinema integrator of Massachusetts Institute. Bull. Acad. Sci. URSS. Cl. Sci. Tech. [Izvestia Akad. Nauk SSSR] **1947**, 645–654 (1947). (Russian)

The instrument was described by Hazen and Brown [J. Franklin Inst. **230**, 19–44, 183–205 (1940); these Rev. **2**, 62.]

Cunningham, W. J. Graphical methods for evaluating Fourier integrals. J. Appl. Phys. **18**, 656–664 (1947).

In many cases the Fourier transform method of determining the response of a linear system to an arbitrary input signal is too complicated for a simple analytic solution, or the data are available only by measurements, thus in form of an empirical curve. To overcome these difficulties the author describes three graphical methods each of which decomposes the functions into a sum of simpler functions with known transforms. Approximate results may quickly be obtained by these methods. *E. Bodewig* (The Hague).

Lemaître, G. Interpolation dans la méthode de Runge-Kutta. Ann. Soc. Sci. Bruxelles. Sér. I. **61**, 106–111 (1947).

The author develops formulae for the numerical solutions $y_i(x)$ $(i=1, \cdots, r)$ of a first order system of the form

$$dy_i/dx = f_i(x; y_1, \cdots, y_r) \tag{1}$$

over the interval $0 \leqq x \leqq 2w$. Whilst the well-known Runge-Kutta formulae yield $y_i(2w)$ from $y_i(0)$ numerically to order $O(w^4)$ the author's interpolation formulae yield $y_i(x)$ to order $O(w^3)$ for any x in the interval $0 \leqq x \leqq 2w$. As with the Runge-Kutta method it is necessary to evaluate the f_i for four sets of arguments x_k, y_{ki}, $k=1, 2, 3, 4$, recurrently defined. Similar formulae are developed for a second order system of the form

$$d^2y_i/dx^2 = g_i(x; y_1, \cdots, y_r) \tag{2}$$

yielding $y_i(x)$ and $dy(x)/dx$ for $0 \leqq x \leqq 2w$ from their initial values at $x=0$. In contrast with finite difference formulae for the solution of (1) and (2) the author's formulae fail to indicate a prescribed decimal accuracy in the y_i unless the expansion coefficients in the f_i or g_i are known numerically.

H. O. Hartley (London).

***Collatz, L. Lösung gewisser Differentialgleichungen mit dem harmonischen Analysator.** Ber. Math.-Tagung Tübingen 1946, pp. 60–61 (1947).

A linear differential equation of nth order with constant coefficients and right-hand member a given function may be solved by means of Fourier integrals, with the aid of a harmonic analyser to evaluate the integrals occurring in the course of the solution. *W. E. Milne* (Corvallis, Ore.).

Kramer, O. P. An application of S. A. Kazakov's method of numerical integration of certain ordinary differential equations. Bull. Acad. Sci. URSS. Cl. Sci. Tech. [Izvestia Akad. Nauk SSSR] **1947**, 609–644 (1947). (Russian)

Kazakov's method is applied to the numerical solution of the equations of exterior ballistics in the form $du/dt = -ku$, $dw/dt = -kw - g$, $dx/dt = u$, $dy/dt = w$, in which k is a function of velocity v and of the altitude y. The idea is to solve the above equations as ordinary linear differential equations treating the function k as though it were a known function of the time t. The actual numerical work is carried out by a step-by-step process using equally spaced values of the time. For the first three steps, the quantity $e^{-\alpha}$, where $\alpha = \int_{t_i}^{t} k\,dt$, is represented by a power series, and formulas are given for the computation of the coefficients. For further steps the procedure for finding $e^{-\alpha}$ is essentially the integration of an extrapolated polynomial with subsequent corrections. An elaborate chart of formulas is provided for carrying out these steps. The method is illustrated by the numerical solution of a problem in ballistics. *W. E. Milne.*

Akušskiĭ, I. Ya. The process of diagonal summation on a tabulator and some of its applications. Bull. Acad. Sci. URSS. Cl. Sci. Tech. [Izvestia Akad. Nauk SSSR] **1947**, 475–495 (1947). (Russian)

The author discusses in great detail a special use of the punched card tabulator and its application to the solution of ordinary differential equations $y' = f(x, y)$, subtabulation, and the evaluation of determinants by means of logarithms. The tabulator described as a "D-11" appears to be equivalent to the model 405. At any rate it is equipped with a number of selectors by which the tabulator may be programmed to punch or record the sum $\epsilon_1 N_1^1 + \cdots + \epsilon_r N_r^r$ $(\epsilon_k^2 = 1)$, where N_i^j is the number punched in the ith field of the jth card. This process is called by the author diagonal summation. The treatment is considerably more mathematical than is usual in punch card literature. [See also same Bull. Cl. Sci. Tech. **1946**, 1081–1120; these Rev. **8**, 171.]

D. H. Lehmer (Berkeley, Calif.).

Akushsky, I. J. The four-counter scheme of solution of Dirichlet's problem by means of punched-card machines. C. R. (Doklady) Acad. Sci. URSS (N.S.) **54**, 659–662 (1946).

Akushsky, I. J. On numerical solution of Dirichlet problem on punched-card machines. C. R. (Doklady) Acad. Sci. URSS (N.S.) **54**, 755–758 (1946).

The author continues [cf. same C. R. (N.S.) **52**, 375–378 (1946); these Rev. **8**, 288] the description of methods for using standard punched card machines for iterating the Liebmann formula for the solution of Dirichlet's problem. Two methods are given for simplifying the procedure and reducing the number of cards. All punched card methods are, however, subject to the defect that there is paralleling of iterations and hence convergence is slower than with hand computation or sequence controlled machines.

P. W. Ketchum (Urbana, Ill.).

Lusternik, L. A., and Prokhorov, A. M. The determination of eigenvalues and eigenfunctions of certain operators by means of a recurrent circuit. C. R. (Doklady) Acad. Sci. URSS (N.S.) **55**, 575–578 (1947).

Various operators, for example those approximating Fredholm symmetric operators, can be represented by the matrix A of a recurrent R, C (resistance, capacitance) network. To determine the eigenvalues $\lambda_1 < \lambda_2 < \cdots < \lambda_n$ of such a matrix with eigenvectors $y_1, y_2, \cdots, y_n$, one can exploit the fact that the vector differential equation $Ay = k\dot{y}$ has the solution $y(t) = \sum_{i=1}^{n} c_i y_i^{-\lambda_i t/k}$, which is dominated for large t by $c_1 y_1 e^{-\lambda_1 t/k}$. By exciting the network nodes with short pulses of period substantially exceeding $1/\lambda_1$ the function $y(t)$ can be displayed on a cathode ray oscillograph and by suitable biasing and time-marker schemes the time-constant $1/\lambda_1$ can be measured to an accuracy of about 3 to 5 per cent. To determine λ_2 the same procedure can be followed except that the initial values at the network nodes must be the components of a vector orthogonal to the first eigenvector; this method can be extended to the other eigenvalues.

H. Wallman (Cambridge, Mass.).

Lusternik, L. A., and Prokhorov, A. M. An experimental determination of the eigenvalues and functions of certain operators by means of an electric RC-circuit. Bull. Acad. Sci. URSS. Sér. Phys. [Izvestia Akad. Nauk SSSR] **11**, 141–145 (1947). (Russian. English summary)

A translation is reviewed above.

Panov, D. Sur la solution approchée des problèmes limites pour les équations non linéaires aux dérivées partielles. C. R. (Doklady) Acad. Sci. URSS (N.S.) **55**, 13–15 (1947).

One method for the approximate solution of boundary-value problems associated with partial differential equations is to replace the differential equation by a system of equations in finite differences. If the differential equation is nonlinear the system of difference equations is also nonlinear. The author applies Newton's method of successive approximations for the solution of such equations provided certain conditions are satisfied. *W. E. Milne* (Corvallis, Ore.).

Fox, L. Some improvements in the use of relaxation methods for the solution of ordinary and partial differential equations. Proc. Roy. Soc. London. Ser. A. **190**, 31–59 (1947).

A common method of solution of boundary-value problems associated with ordinary or partial linear differential equations is to replace the differential equation by difference equations which are then solved by successive approximations. The usual practice, especially in the case of partial differential equations, is to replace derivatives by the leading term only in the finite difference series for the derivative, and then to take the interval small enough to secure the desired accuracy. The author proposes to employ differences of higher orders so as to secure the required accuracy with a larger interval and hence with a smaller number of unknowns to be determined.

Two worked examples illustrate the procedure for ordinary second order differential equations with assigned values at the ends of the interval. For the case of partial differential equations, Poisson's equation with assigned boundary values is solved for a square region using rectangular coordinates, and for a cylinder using cylindrical coordinates. To illustrate the determination of characteristic numbers (eigenvalues) the frequencies of two modes of vibration for a square membrane are obtained. Two further examples show how the method may be applied in the case of curved boundaries where points of the mesh do not fall on the boundary. *W. E. Milne* (Corvallis, Ore.).

Bowie, O. L. A least-square application to relaxation methods. J. Appl. Phys. **18**, 830–833 (1947).

If $L(u_i)=0$ is the linear difference equation by which a given linear partial differential equation has been replaced, the usual procedure is to solve the set of equations $L(u_i)=0$ by successive approximations. The author proposes to speed the convergence of this process by means of least squares. If $\bar{u}_i$ is the approximate solution of the difference equation and boundary conditions at some stage of the process, and if $\bar{u}_i+cu_i$ satisfy the boundary conditions, the constant c is determined so as to minimize $\sum_i\{L(\bar{u}_i)+cL(u_i)\}^2$. The author discusses the advantages of various ways of choosing the u_i so as to improve convergence. *W. E. Milne.*

Faedo, Sandro. Sul metodo di Ritz e su quelli fondati sul principio dei minimi quadrati per la risoluzione approssimata dei problemi della fisica matematica. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **6**, 73–94 (1947).

Ritz's method for solving the problems of mathematical physics is compared with that of Picone, based on the method of least squares. [Cf., for instance, M. Picone, Rend. Circ. Mat. Palermo **52**, 225–253 (1928).] These methods are intimately related. A solution is sought of the differential equation $L[y]=f(x)$, satisfying given homogeneous boundary conditions for $x=a$ and $x=b$. In both methods there occurs a chosen infinite system of functions $\varphi_k(x)$. If the coefficients of $L[y]$ and $f(x)$ have continuous derivatives up to order $2+\nu$ ($\nu>0$), and 1 is not an eigenvalue, then choosing, for the system $\varphi_k(x)$,

$$\{\cos[2\pi k(x-a)/(b+\delta-a)],\quad \sin[2\pi k(x-a)/(b+\delta-a)]\},$$

Picone has proved in the paper quoted above that the error in his nth approximation is of the order $n^{-1-\nu}$, and so, if the coefficients and $f(x)$ are analytic, the error is infinitesimal of arbitrarily high order.

In using Ritz's method it is natural to choose, for the system $\varphi_k(x)$, $\sin[k\pi(x-a)/(b-a)]$, but then it can only be deduced that the error in the nth approximation is of the order n^{-1} at most, even if $f(x)$ and the coefficients of $L[y]$ are analytic. This seems a serious disadvantage of Ritz's method as compared to Picone's. Now the author proves (1) that Picone's theorem holds also if, instead of the system φ_k mentioned above, there is chosen simply the system $1, x, x^2, \cdots$, but it fails if in the system above δ is taken to be 0; (2) that an analogous theorem is valid for Ritz's method, if the system chosen is:

$$\varphi_k(x)=(x-a)(b-x)\sin[k\pi(x-a+\delta)/(b+2\delta-a)],\qquad \delta>0,$$

or also $\varphi_k(x)=(x-a)(b-x)x^k$, and some others. It is assumed that the least eigenvalue is greater than 1. *H. Bremekamp* (Delft).

Vilner, I. A. On the nomograms of elliptical functions and integrals in the complex domain. C. R. (Doklady) Acad. Sci. URSS (N.S.) **55**, 783–786 (1947).

This is a continuation of a previous paper [same C. R. (N.S.) **53**, 187–190 (1946); these Rev. **8**, 494] in which the author studies the representation of elliptic integrals by nomographs with two rectilinear scales. *P. W. Ketchum* (Urbana, Ill.).

Lah, Ivo. Das Zinsfussproblem. Mitt. Verein. Schweiz. Versich.-Math. **47**, 167–247 (1947).

If the commutation columns for one or more rates of interest are given it is possible to compute the values of actuarial functions for these rates of interest. It is sometimes desirable to find an approximation of the value of an actuarial function for a rate of interest for which the commutation tables are not available. The problem of deriving approximations, using only the available commutation columns, is called the problem of interest in actuarial mathematics.

The author surveys systematically different solutions of this problem. He generalizes the formula of K. Poukka [Skand. Aktuarietidskr. **6**, 137–152 (1923)]. The subsequent discussion is based on this generalization. The solutions of the problem of interest are classified into four groups. (1) Formulae using commutation columns for a single rate of interest. (2) Formulae using commutation columns for several rates of interest. These are essentially methods of interpolation or extrapolation. (3) Formulae assuming an analytic law of mortality. (4) Formulae using annuities certain. *E. Lukacs* (Cincinnati, Ohio).

Perks, Wilfred. Two-variable developments of the n-ages method. J. Inst. Actuar. **72**, 377–397; discussion, 398–414 (1946).

If the values of the life assurances of an office are required on different technical bases, it is of great importance to avoid combining the elements of the different bases with the single policies. The author gives a number of quadrature formulae expressing the total value in a small number (most often 4 or 9) of terms, each of them being the product of one factor depending only on the valuation bases and the other on the distribution of the sums assured. In numerical examples from well-known model offices very good approximations are found. *P. Johansen* (Copenhagen).

Joseph, A. W. The valuation of whole-life assurances by the use of moments. J. Inst. Actuar. **72**, 498–515 (1946).

The methods of Perks's paper reviewed above are developed and tested on the valuation of the whole-life assurances of a life office. *P. Johansen* (Copenhagen).

Wilhelmsen, Lars. On the valuation of life policies. Skand. Aktuarietidskr. **30**, 8–17 (1947).

Renberg, A. Une méthode pour calculer les réserves mathématiques à l'inventaire. Skand. Aktuarietidskr. **30**, 1–7 (1947).

RELATIVITY

Hill, E. L. On the kinematics of uniformly accelerated motions and classical electromagnetic theory. Physical Rev. (2) **72**, 143–149 (1947).

In continuation of earlier work by the author [see the same Rev. **67**, 358–363 (1945); these Rev. **7**, 88] this paper is concerned with the 15-parameter group of conformal transformations in 4-dimensional space-time. This group is characterised by a line-element of the form $ds^2=\lambda^2(d\tau^2-dx^2-dy^2-dz^2)$ and a subgroup is the 10-parameter group of Lorentz transformations. The motion of a particle is examined and it is found that transformations of the conformal group are associated with uniformly accelerated motions. The implications for Dirac's classical-type electron theory are discussed. *A. G. Walker* (Sheffield).

Hill, E. L. The relativistic clock problem. Physical Rev. (2) **72**, 236–240 (1947).

The clock problem of relativity theory (Einstein's "clock paradox") is examined in relation to the author's theory of uniformly accelerated motions in which the Lorentz group of transformations is replaced by the conformal group in space-time [see the preceding review]. Two solutions are found giving the desired relative motion, and in each case it appears that the "paradox" no longer exists, two clocks which agree when they part company being found to agree when they are again coincident. The bearing of this problem on the relation between mechanics and electromagnetic theory is discussed briefly. *A. G. Walker* (Sheffield).

Ives, Herbert E. The physical significance of Birkhoff's gravitational equations. Physical Rev. (2) **72**, 229–232 (1947).

The author begins with the remark that, in Birkhoff's theory [Revista Ci., Lima **44**, 253–257 (1942); Proc. Nat. Acad. Sci. U. S. A. **29**, 231–239 (1943); **30**, 324–334 (1944); these Rev. **4**, 116, 285; **6**, 72], stress is laid on the use of local time ds in place of Newtonian time dt, and that Minkowskian forces are used in place of physical (Lorentzian) forces. Yet Birkhoff's solution for a planetary orbit, which gives the correct advance of perihelion of Mercury, contains no terms in ds, and the question therefore arises whether the result does in fact depend upon the use of this variable. To obtain an answer, the author transforms Birkhoff's gravitational equations for a particle in the field of a central mass into equations involving dt. He finds that, although Birkhoff regarded local time as of basic significance, the theory could have been developed as easily, and with greater physical significance, if it had been founded instead upon the idea of local (as distinct from stationary) mass. He also finds that the Birkhoff force equations for a planetary orbit are those one would obtain from a Newtonian attractive force acting on the Lorentzian local mass of the planet, with the addition of forces caused by motion, transverse to the path of the planet, which do not affect the conservation of energy but alter the areal constant $r^2\dot{\theta}$. The terms representing these forces are, in the author's view, the key contribution of Birkhoff's theory, though at present they lack physical arguments in support of them. *H. S. Ruse* (Leeds).

Gião, Antonio. Sur le magnétisme des masses en rotation. C. R. Acad. Sci. Paris **224**, 1813–1815 (1947).

The relation $M_{\text{magn.}}=(\beta\sqrt{K}/2c)M_{\text{rot.}}$ (K the constant of gravitation, c the speed of light, β a numerical constant about $\frac{1}{4}$), connecting the angular momentum of rotation and the magnetic moment sufficient to produce the observed magnetic fields, has been shown to be satisfied for the earth, the sun and the star 78 Virginis and has been suggested to be a general relation [Blackett, Nature **159**, 658–666 (1947)]. The present author states that for a nonelectrified uniformly rotating spherical mass the coefficients g_{ik}, ω_{ik} of the first and second fundamental forms (his "internal" and "external" metrics to which, respectively, he relates gravitational and electromagnetic phenomena) satisfy $\omega_{4i}=\text{constant}\cdot g_{4i}$ ($i=1, 2, 3$). The values of g_{4i} are those given by Lense and Thirring [Phys. Z. **19**, 156–163 (1918)]. From this relation and from his previous work [Portugaliae Phys. **2**, 1–98 (1946); these Rev. **8**, 121] he deduces a connexion of the above type between magnetic moment and angular momentum. *C. Strachan* (Aberdeen).

Corben, H. C., and Corben, Mulaika. The motion of a particle in an electromagnetic and gravitational field. Physical Rev. (2) **72**, 434 (1947).

It is shown that the equations of motion for a particle in an electromagnetic and gravitational field may be derived from a simple Lagrangian which is based on an extended principle of special relativity theory. In terms of coordinates q^α ($\alpha=1, \cdots, 8$) of position and velocity space, the Lagrangian is $L=A_\alpha dq^\alpha/d\tau$, where $d\tau=(dq_\alpha dq^\alpha)^{\frac{1}{2}}$ and A_α is an 8-vector describing the interaction between the particle and the field. The authors discuss the case when the A_α depend only upon the position coordinates in 4-space; the general case is left to a later paper. *A. G. Walker*.

Hoffmann, Banesh. The vector meson field and projective relativity. Physical Rev. (2) **72**, 458–465 (1947).

This paper is concerned with the formal derivation of the field equations of general relativity in a 5-dimensional projective space. Writing $G_{\alpha\beta}$ ($\alpha, \beta=0, 1, \cdots, 5$) for the projective metric tensor, of index $2N$, in 5-space, it has been usual to consider variations of the restricted metric tensor $\gamma_{\alpha\beta}=G_{\alpha\beta}/G_{00}$, of index zero, and to obtain the field equations from the variational equation $\delta\int B\gamma^{\frac{1}{2}}dx^1dx^2dx^3dx^4=0$, where B is the scalar curvature formed from $\gamma_{\alpha\beta}$. In the present paper this variational equation is replaced by one in which the integral is still quadruple but the integrand is $\bar{P}G^{\frac{1}{2}}\Phi^{-2}$, where $\Phi=(G_{00})^{\frac{1}{2}}$ and $\bar{P}$ is the scalar curvature formed from $G_{\alpha\beta}$ in 5-space. Except for one term the resulting equations

are shown to have the form of the unquantized field equations for a vector meson and gravitational field in the general theory of relativity. The extra term is found to be too large to be interpreted as a cosmological term and can be removed by a modification of the integrand in the variational equation. It is suggested that an extension to conformal geometry would prove of interest. *A. G. Walker.*

Dive, Pierre. L'électro-optique dans le temps universel. Bull. Astr. (2) **12**, 1–71 (1940).

MECHANICS

Pírko, Zdeněk. Theory of gliding motion. Věstník Královské České Společnosti Nauk. Třída Matemat.-Přírodověd. **1946**, no. 16, 9 pp. (1947). (Czech)

Let a Cartesian coordinate system $\Sigma(\xi, \eta)$ glide along a curve C so that the origin A of Σ moves along C, the ξ-axis is always tangential and the η-axis normal to the curve. The author studies the motion of Σ analytically, by means of the natural equation (the relation between radius of curvature and length of arc) of C. First the author studies the trajectory Γ of a point P whose position is fixed with respect to Σ. He proves that the normal of Γ at P passes through the corresponding center of curvature S of C and that Γ is the envelope of the circle with centre S radius SP; he also obtains expressions for the length of arc and for the radius of curvature of Γ. These preparations enable him to solve kinematical problems such as how to choose C so as to make the motion of a certain point P a translation or a rotation, or how to choose C so as to make Γ congruent with C. In particular, if P is on the η-axis Γ is a parallel curve to C, and if P is on the ξ-axis Γ is an equitangential curve of C.

In a similar manner the envelope E of a straight line Π (fixed with respect to Σ) is investigated and the following kinematical problems are solved (B is the point of contact of E and Π): how to choose C so that (i) the velocities of A and B should bear a constant ratio, (ii) B's motion should be a rotation and (iii) E should be congruent to C. Particular positions of Π correspond to parallel curves to C, to the evolute and to the evolutoids of C. *A. Erdélyi.*

Bruevič, N. G. On the error in the velocity ratio of a plane cam mechanism. Bull. Acad. Sci. URSS. Cl. Sci. Tech. [Izvestia Akad. Nauk SSSR] **1947**, 497–502 (1947). (Russian)

A cam, which may be pivoted or sliding, can drive another cam to produce desired motions. The instantaneous ratio of the motions of the cams, usually called the speed ratio, is a function of the configuration and the curves of the cams. In manufacture, however, the actual cams may deviate from the design to produce departures from the desired motions. The author derives several formulas, using elementary differential geometry, giving the errors in the desired speed ratio in terms of the parameters of the configuration and the deviations of the radii of curvature of the cam surfaces from the design curvatures. *M. Goldberg.*

Byhovskiĭ, M. L. On a method for determining the errors of velocity and acceleration of plane mechanisms. Bull. Acad. Sci. URSS. Cl. Sci. Tech. [Izvestia Akad. Nauk SSSR] **1947**, 503–510 (1947). (Russian)

The errors in any mechanism, due to unavoidable deviations from design in manufacture, are expressible approximately in terms of the design parameters and the first order deviations. The author describes a graphical method for obtaining errors in speed ratios and accelerations. He employs the expedient of treating deviations as deformations of the members which, in turn, produce displacements in the other members of the mechanism. Therefore, for each position of the given mechanism, a new mechanism is constructed whose motions correspond to errors produced by deformations of the members of the given mechanism. By combining the separate errors, an estimate of the total error is obtained. *M. Goldberg* (Washington, D. C.).

★Rosser, J. Barkley, Newton, Robert R., and Gross, George L. Mathematical Theory of Rocket Flight. McGraw-Hill Book Company, Inc., New York, 1947. viii+276 pp. $4.50.

This book, which contains the only serious treatment of its subject in print, is the official final report to the Office of Scientific Research and Development concerning the work done on the exterior ballistics of fin-stabilized rockets at the Allegany Ballistics Laboratory during 1944 and 1945 when the laboratory was operated by The George Washington University. The theory described has been built up as a result of the labours of many different scientists and mathematicians, not only in the United States but also in Great Britain where the theory of rocket motion was well advanced even before 1939. Due acknowledgment is paid by the authors to all who have helped, although secrecy restrictions preclude the mention of individual contributions.

The book contains five chapters, five appendices, a bibliography, a glossary of symbols and an index. In chapter I the forces due to the action of the gas jet, the air stream and gravity are described and the equations of motion are formed. The feature which distinguishes the mathematical theory from that appropriate to other projectiles is the fact that the mass does not remain constant nor does the centre of gravity remain fixed in position relative to the metal frame. For this reason the principles of the conservation of linear and angular momentum have to be applied with care. This the authors do by setting up five principles which are valid for systems of changing mass content. The equations of motion are obtained in a two-dimensional form. This is possible since only unrotated motion is considered, and it has the advantage of being easily understood and of not requiring the reader to visualise in three dimensions. On the other hand, the equations are afterwards modified (by the usual device of introducing complex quantities) to cater for three-dimensional motion and a slow axial spin. This is quite legitimate in the cases the authors consider (namely, spin low enough to avoid gyroscopic effects) but can really only be convincing to a reader who retraces his steps and forms the equations afresh in the three-dimensional form. If the authors had adopted this course the treatment would have been considerably more complicated. On the other hand certain parts of the theory would have been clarified. For example, the correct interpretation of the term $-K_{S}\rho d^3 v\dot{\varphi}\delta$, whose meaning in terms of complex yaw and cross spin is discussed on page 48, would then have been obvious.

Chapter II is devoted to the motion after the burning of the propellant has ceased. Due to asymmetries of design and functioning, such as bent fins and malaligned thrust, the initial conditions at the beginning of this part of the trajectory will be different for each rocket fired. These initial conditions, their effect upon the ensuing motion and

the continuing effect of certain asymmetries (e.g., bent fins) cause the points of impact or burst of a group of rounds fired under similar conditions to form a scattered pattern. The accuracy of the weapon can be assessed from estimates of the dispersion of this scatter. The subject of dispersion is discussed fully and is based upon estimates of the deviation of the round from the idealized trajectory which would be followed by a particle subject only to gravity and drag. This deviation is termed dispersion throughout, but the reviewer found it difficult to decide whether the authors attach any statistical significance to this term, since although the chief part of the deviation (namely, that due to the asymmetries mentioned above) is of a random character, it contains contributions (such as those due to the effect of the increased drag due to yaw, and of the lift force in conjunction with the yaw of repose) which, although demonstrably negligible for the cases considered, are systematic and not random.

The most important part of the book is chapter III in which the motion during burning is considered. The equations of motion are solved under general conditions for an unrotated, or very slowly rotating, rocket subject to the asymmetries mentioned. These solutions, which hold under various assumptions, such as constant acceleration and yaw wave-length and no damping, are expressed in terms of certain "rocket functions" which are fully described and tabulated in chapter V. These functions can be expressed in terms of the Fresnel integrals and their integrals. The latter being highly oscillatory are unsuitable for tabulation and interpolation, but the rocket functions do not suffer from this defect since they are monotonic. The two basic functions $rr(x)$ and $rj(x)$ are essentially the functions $(K\pm H)/\pi^{\frac{1}{2}}$ first introduced and tabulated by Miller and Gordon [J. Phys. Chem. **35**, 2785–2884 (1931)]. Useful approximate solutions of the equations are given and the solutions under more general assumptions are also discussed. In this connection it is worth remarking that, in the case where the yaw wave-length is assumed to be practically constant, the equations can be solved in closed form for a variable acceleration of any type so that it is possible by this means to assess the effect of variable thrust more satisfactorily than is indicated in § 10 of the chapter. Of the other subjects discussed in chapter III special mention may be made of the section dealing with the fire from aircraft of rounds with large initial yaw. The chapter closes with a well balanced discussion of dispersion as predicted by the theory from estimates of various malalignments and as observed in practice.

In chapter IV the effect of tip-off from the launcher (projector) is considered for two types of launcher and for flexible and rigid launchers. Of the appendices the most important is that which describes the one-dimensional theory of flow through the nozzle and gives formulae for the efflux velocity in terms of the nozzle throat area and other parameters.

The authors state that they have two types of reader in view, namely (i) the trained scientist with no previous experience of rockets and (ii) the person with little scientific training who is interested in what makes a rocket go. Both types should find much in the book to attract them; it is, however, the professional rocket ballistician who will be benefited most by having at hand this comprehensive treatise on his subject. At each stage of the work the reader's feet are kept firmly on the ground by numerical examples based on actual rockets. *R. A. Rankin* (Cambridge, England).

Signorini, Antonio. Complementi alla dinamica dei giroscopi e equazioni del problema completo della balistica esterna. Atti Accad. Naz. Lincei. Mem. Cl. Sci. Fis. Mat. Nat. (8) **1**, 1–41 (1946).

This paper deals with the problem of external ballistics in a form which is complete in the sense that no effects which are appreciable in practice are neglected. After a preliminary development of the differential equations of motion of a rigid body, the differential equations of the ballistic problem are written in a form which is considered to be particularly convenient for the solution by perturbation methods. As the basis for such a solution the author chooses the solution of a restricted problem, which he discusses at considerable length. In the case of the restricted problem the solution of the differential equations reduces to the solution of a system of the fourth order and quadratures. A rather full discussion of the local properties of the trajectory in the restricted problem is given. None of the proposed perturbation theory is developed; the study of the trajectory in the complete problem is confined to the investigation of a few simple properties of the trajectory in the neighborhood of the initial point. *L. A. MacColl.*

Leimanis, Eugène. Sur l'intégration par quadratures des équations du mouvement d'un projectile dans un milieu de densité variable. C. R. Acad. Sci. Paris **224**, 1618–1620 (1947).

Leimanis, Eugène. Sur l'intégration par quadratures des équations du mouvement d'un projectile dans un milieu de densité et température variables. C. R. Acad. Sci. Paris **224**, 1752–1754 (1947).

The author considers briefly the differential equations of the exterior ballistics of a particle, using an air density distribution varying exponentially with altitude; in the first article air temperature is regarded as constant, in the second as decreasing with linear gradient in the altitude. The author examines the equations under the hypothesis that a solution exists independent of the ballistic coefficient. With this unrealistic hypothesis the author succeeds, in a pre-Bernoullian fashion, in obtaining solutions by quadrature, extending previous results by Appell and Drach.

A. A. Bennett (Providence, R. I.).

***Popoff, Kiril. Problème principal de la balistique extérieure.** Atti Convegno Mat. Roma 1942, pp. 133–141 (1945).

***Duncan, W. J. Mechanical admittances and their applications to oscillation problems.** Ministry of Supply [London], Aeronaut. Res. Council, Rep. and Memoranda no. **2000** (Monograph), 128 pp. (1947).

Sâlceanu, Constantin, et Adam, Semlien. Oscillations de deux pendules de résonance. C. R. Acad. Sci. Paris **225**, 102–104 (1947).

Masotti, Arnaldo. Sui moti oscillatorii di un punto. Ist. Lombardo Sci. Lett. Cl. Sci. Mat. Nat. Rend. (3) **4(73)**, 49–61 (1940).

The author studies the motion of a particle on a straight line under the influence of a positional force and investigates the conditions under which the motion is the projection of a circular orbit with acceleration directed towards a fixed center C. This is found to be so when the positional force is inversely proportional to the cube of the distance

As the author properly notes, it is the infinitesimal transformation T, not the trajectory itself, which is assumed to be independent of the ballistic coefficient.

from a fixed point H, which is the polar of C with respect to the circle. *W. Kaplan* (Ann Arbor, Mich.).

Hostinský, B. Sur les oscillations forcées des systèmes mécaniques ou électriques. Acad. Tchèque Sci. Bull. Int. Cl. Sci. Math. Nat. **40**, 139–146 (1939).

This paper is based on a remark by Rayleigh [Theory of Sound, 2d ed., v. 1, Macmillan, London, 1926, p. 74] that the effect of a forcing term on a harmonic oscillator is to add a differential displacement proportional to the change in velocity caused by the forcing term. This point of view is shown to be applicable to vibrating systems of finite or infinite number of degrees of freedom, e.g., the vibrating string, electromagnetic waves. *W. Kaplan.*

Hostinský, Bohuslav. Das akustische Spektrum einer Saite. Acad. Tchèque Sci. Bull. Int. Cl. Sci. Math. Nat. **44**, 23–32 (1943).

The Lagrangian model of a string is used: m mass-particles are equally spaced along a line and allowed to move subject to forces perpendicular to the line. The forces are derived from a potential which is quadratic in the displacements and takes into account interactions between "nearest neighbors." [See Rayleigh, Theory of Sound, 2d ed., v. 1, Macmillan, London, 1926, p. 120.] In addition, the system is assumed to be subject to impulses occurring at random intervals, the corresponding probability distribution being proportional to $\exp(-ct)$, where t is the time and c is constant. The author obtains simple explicit formulae for the mean energy of the system after time t and the distribution of this mean energy among the m eigenfrequences, i.e., the energy spectrum. *W. Kaplan.*

Hostinský, Bohuslav. Über die Verteilung der Energie in akustischen Spektren. Acad. Tchèque Sci. Bull. Int. Cl. Sci. Math. Nat. **44**, 393–398 (1943).

This paper is a continuation of that of the previous review. The system is now assumed subjected to impulses due to collisions of one particle with an external particle moving on the same line and being reflected by a fixed wall. A generalization of this problem to a similar finite model for a membrane is also considered. For both cases an equipartition theorem for the energy spectrum is stated without proof. *W. Kaplan* (Ann Arbor, Mich.).

Bureau, Florent. Sur le calcul de l'effet gyroscopique en mouvement relatif. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **32** (1946), 80–85 (1947).

Angheluţă, Th. Remarques sur des mouvements tautochrones. Bull. Sci. École Polytech. Timişoara **10**, 69–72 (1941).

For a given family of curves in a plane, $f(x, y, a)=0$, the force function $U(x, y)$ which leads to tautochronous motion is determined. The method follows closely that used for special cases in P. Appell's Mécanique Rationelle [vol. 1, 4th ed., Paris, 1919, pp. 351, 460]. *P. Franklin.*

Bădescu, Radu. Sur le mouvement tautochrone plan. Bull. Sci. École Polytech. Timişoara **11**, 81–103 (1943).

Several cases of tautochronous motion in a plane for given individual curves or given force field are discussed. The Poincaré-Bendixon results on singular points of differential equations are applied. *P. Franklin* (Cambridge, Mass.).

Costa de Beauregard, Olivier. Retour sur la théorie du spin et sur la dynamique des systèmes de points. C. R. Acad. Sci. Paris **225**, 523–525 (1947).

Kasner, Edward. Physical curves. Proc. Nat. Acad. Sci. U. S. A. **33**, 246–251 (1947).

This note is concerned with a unification of parts of the author's previous work on the geometrical aspects of dynamics. Suppose that we have a positional field of force in a plane and a given curve in the plane. We consider a particle which is constrained to move along the curve and which traverses a prescribed point with a prescribed speed. When the particle is at the typical point of the curve it is subjected to a certain force N perpendicular to the tangent to the curve and it exerts a certain pressure P upon the curve. There then arises the problem of determining those curves for which we have the relation $P=kN$, where k is a prescribed constant. It is found that the families of curves corresponding to certain values of k are identical with families which the author has considered from other standpoints in his earlier investigations. For $k=0$ we have the family of trajectories of unconstrained motion in the field of force; for $k=-2$ we have the family of pseudo-brachistochrones; for $k=1$ we have the family of generalized catenaries; for $k=\infty$ we have what the author calls the family of velocity curves. A set of geometrical properties is given which characterizes the family of curves corresponding to an arbitrary value of k and some other properties are stated. The note closes with some discussion of the four-parameter family of curves which is the set-theoretic sum of the three-parameter families corresponding to the several possible values of k. *L. A. MacColl* (New York, N. Y.).

De Cicco, John. New proofs of the theorems of Kasner concerning the infinitesimal contact transformations of mechanics. J. Math. Phys. Mass. Inst. Tech. **26**, 104–109 (1947).

The theorems of Kasner which are under consideration relate to two conservative holonomic dynamical systems, with the same number of degrees of freedom, for which the commutator of the associated infinitesimal contact transformations is a point transformation. [See Bull. Amer. Math. Soc. **16**, 408–412 (1910).] The main theorem states that, if two dynamical systems have the stated property, the corresponding expressions for the kinetic energy are the same to within a factor which depends only upon the coordinates. The converse also holds. A second theorem states that two infinitesimal contact transformations with the same transversality law have a point transformation for commutator if, and only if, they are associated with two dynamical systems of the type described above. The new proofs, in which the notation and some of the elementary concepts of the tensor calculus are used, proceed simply and directly from the basic properties of infinitesimal homogeneous contact transformations, and from the relations of these transformations to dynamical systems.

L. A. MacColl (New York, N. Y.).

Kravtchenko, Julien. Sur les équations générales de la dynamique des systèmes. Ann. Univ. Grenoble. Sect. Sci. Math. Phys. (N.S.) **22**, 281–297 (1946).

The author develops an alternative form to the general equations of Appell, which apply to both holonomic and nonholonomic dynamical systems. He shows the advantage of his modified procedure by applying it to some specific problems of rigid dynamics. *P. Franklin.*

Aminov, M. S. On the equation of disturbed motion of a mechanical system. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] 11, 377–378 (1947). (Russian. English summary)

The author derives the general equations of the disturbed motion for a conservative holonomic system. He uses the action line-element $ds^2=4T^2dt^2$ of Synge [Philos. Trans. Roy. Soc. London. Ser. A. 226, 31–106 (1926)]. Using a suitable coordinate system and correspondence between the points of the undisturbed and the disturbed motions, the equations attain a particularly convenient form.

J. Lifshitz (México, D. F.).

***Hamel, G. Aus der analytischen Mechanik.** Ber. Math.-Tagung Tübingen 1946, pp. 72–73 (1947).

The nonholonomic system afforded by a heavy sphere, with its center of gravity G not at the geometrical center, rolling on a rough plane is considered for various mass distributions. The limiting case when all of the mass is concentrated at G yields a problem in particle dynamics. From a consideration of a first integral in the general case, the author is led to make the following statement: "Ein Problem der Punktmechanik hängt also in seinem Ergebnis wesentlich von dem Grenzübergang zum Punkt ab." The paper also contains a note on a possible form for the equations of a top.

D. C. Lewis (College Park, Md.).

Pignedoli, Antonio. Sui sistemi lagrangiani con forze dipendenti dalle accelerazioni. Atti Soc. Nat. Mat. Modena (6) 77, 100–109 (1946).

This paper sets forth some formal facts regarding ignoration of coordinates for dynamical systems in which the Lagrangian forces depend linearly on the accelerations and quadratically on the velocities. The electrodynamical equations of Weber for the motion of two charged particles are adduced as an illustration.

D. C. Lewis.

Agostinelli, Cataldo. Sull'esistenza di integrali di un sistema anolonomo con coordinate ignorabili. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. 80, 231–239 (1945).

The author seeks to generalize to nonholonomic systems the known momentum integral corresponding to an ignorable coordinate in a holonomic system. This is shown first to be possible if the restraints are independent of both the ignorable coordinate and its time-derivative. Other more general and more complicated conditions are given under which the integral exists.

W. Kaplan (Ann Arbor, Mich.).

Castoldi, Luigi. Sopra una classe di sistemi dinamici soggetti a vincoli di mobilità per cui si annullano i corrispondenti termini di anolonomia. Boll. Un. Mat. Ital. (3) 2, 19–23 (1947).

Under certain very restrictive conditions the equations of a nonholonomic system can be written in Lagrangian form. In this note a simple example of such a system is given.

D. C. Lewis (College Park, Md.).

Hydrodynamics, Aerodynamics, Acoustics

Pólya, G. A minimum problem about the motion of a solid through a fluid. Proc. Nat. Acad. Sci. U. S. A. 33, 218–221 (1947).

When a solid moves with velocity U in uniform incompressible inviscid fluid otherwise at rest, the kinetic energy of the fluid is $\frac{1}{2}MU^2$. Here M is called by the author the virtual mass [but the term is often applied to M_0+M where M_0 is the mass of the solid]. The value of M depends, in general, on the direction of U relative to a frame of reference fixed in the solid. The average virtual mass is then defined as the average of M over all orientations of U with respect to the solid. It can be proved that of all ellipsoids with given volume the sphere has the least average virtual mass. The author conjectures that this theorem will remain true if for "ellipsoids" the word "solids" is substituted. This paper takes the first step to confirming the conjecture by proving, by mapping on the unit circle, that, in two-dimensional motion, of all cylinders having the same area of cross-section the circular cylinder has the least average virtual mass per unit height.

L. M. Milne-Thomson.

***Liepmann, Hans Wolfgang, and Puckett, Allen E. Introduction to Aerodynamics of a Compressible Fluid.** John Wiley & Sons, Inc., New York, 1947. ix+262 pp. (2 plates). $4.00.

This is an introduction to gas dynamics written primarily for engineers and stressing those parts of the theory which lead to practical applications. About one third of the book develops one-dimensional gas dynamics and deals almost exclusively with steady motions. The basic thermodynamical and mechanical concepts are developed in the first two chapters. Then channel flows and shocks are considered. Supersonic wind tunnel design and observation methods are described. The second part deals with steady flows in two and three dimensions. After the derivation of the fundamental equations the following subjects are discussed: small perturbation methods, the Rayleigh-Janzen method, the hodograph method (almost exclusively from the point of view of the Chaplygin approximation as used by von Kármán and Tsien), the Prandtl-Meyer flow, the exact solution derived by Ringleb (and in this connection the concept of the limiting line), and the Prandtl-Busemann method for two dimensions. Two brief chapters on viscosity effects and on the comparison between theory and experiments on airfoils conclude the book.

The exposition is very careful and clear and a mathematician intending to work in theoretical gas dynamics will profit greatly from reading this book. He will have to turn elsewhere, however, for the deeper mathematical problems occurring in this field. The authors give only a brief table of references which includes very few recent mathematical papers.

L. Bers (Syracuse, N. Y.).

von Kármán, Theodore. Supersonic aerodynamics—principles and applications. J. Aeronaut. Sci. 14, 373–402 (1947).

The tenth Wright Brothers Lecture.

***Emmons, Howard W. Gas Dynamics Tables for Air.** Dover Publications, Inc., New York, 1947. 46 pp. $1.75.

Cârstoiu, I. Courbes et surfaces correspondantes dans la déformation d'un milieu. Bull. Sci. École Polytech. Timişoara 12, 189–193 (1946).

Consider a medium which admits a deformation so that the point with coordinates (x_0^λ) goes into the point with coordinates (x^λ). Let the vector field defined by $dx_0^\lambda=\alpha X_0^\lambda$ be transformed into the vector field $dx^\lambda=\beta X^\lambda$. If $x^\lambda=x^\lambda(x_0^\mu)$ is the transformation taking (x_0^λ) into (x^λ) then

(1) $\beta X^{\lambda}=\alpha X_0^{\mu}(\partial x^{\lambda}/\partial x_0^{\mu})$. Appell has shown [J. Math. Pures Appl. (5) 5, 137–153 (1899)] that if $\beta/\alpha=\Delta=|\partial x^{\lambda}/\partial x_0^{\mu}|$ then (2) $\partial X_0^{\mu}/\partial x_0^{\mu}=\Delta(\partial X^{\mu}/\partial x^{\mu})$. Hence, if X_0^{μ} is the velocity field of an incompressible fluid flow then X^{μ} is another velocity field of an incompressible flow. By introducing proper factors in (1), this result can be extended to compressible flows. The author extends Appell's results by finding: (1) the relation between linear elements of the above vector fields; (2) a surface invariant; (3) a special case in which the surface S_0 and the transformed surface S are applicable. *N. Coburn* (Ann Arbor, Mich.).

Cârstoiu, Ion. Sur la possibilité des mouvements irrotationnels d'un fluide visqueux incompressible. C. R. Acad. Sci. Paris **225**, 664–666 (1947).

Following the considerations concerning the existence of an irrotational flow of an incompressible viscous fluid by G. Hamel [Z. Angew. Math. Mech. **21**, 129–139 (1941); these Rev. **3**, 92] and the existence of a potential function by H. Jeffreys [Proc. Cambridge Philos. Soc. **24**, 477–479 (1928)], the author extends his earlier work [same C. R. **223**, 1095–1096 (1946); these Rev. **8**, 294] on the distribution of vortices at any time to the existence of the velocity and the pressure. The formulas obtained show precisely when the potential flow breaks down. They are similar to the formulas obtained by L. Lichtenstein [Grundlagen der Hydromechanik, Berlin, 1929, pp. 409–411] for a perfect fluid. *A. Gelbart* (Princeton, N. J.).

Jacob, Caius. Considérations élémentaires sur la double-source. Disquisit. Math. Phys. **1**, 369–390 (1941).

A study is made of the geometric features of the dipole flow of a gas. By use of the exact solutions given by Chaplygin's method, some results are obtained which differ from those of Ringleb's approximate solution of the same flow problem. *D. Gilbarg* (Bloomington, Ind.).

Burgers, J. M. Cases of motion in a gas with noncolliding molecules. Nederl. Akad. Wetensch., Proc. **50**, 573–583 (1947).

The problem considered is to calculate the velocity distribution function of molecules between two parallel walls, of infinite extent, or of molecules on one side of a single wall under the assumptions that (1) the velocity of the wall V is normal to the plane of the wall in the direction of the X-axis; (2) the initial distribution before the motion of the wall is specified; (3) the internal energy of the molecules is zero; (4) the kinetic energy relative to the wall carried by the molecules is not changed by collision with the wall; (5) the distribution function for reflected molecules from the wall is given by $F_{\text{refl.}}(t; u, v, w)=A(h/\pi)^{\frac{3}{2}}e^{-h\{(u-V)^2+v^2+w^2\}}$, where v and w can take all values, while $u\geqq V$ for the wall on the left of the gas; (6) the collisions among the molecules can be neglected. It is seen that the only conditions for determining the distribution function are (2) and (4). Two cases are treated in some detail. (a) The case of a single wall moving with high velocity and zero initial molecular velocity. Here considerable deviation from the Maxwellian distribution is found. (b) The expansion of rarefied gas between two symmetrically receding walls. Here only the first steps of solution could be taken for the special case where the velocity of the walls is small compared to that of the molecules. Then the temperature of the gas is found to decrease in a way predicted by Poisson's law for adiabatic change of state. *H. S. Tsien* (Cambridge, Mass.).

Hicks, Bruce L. On explicit solutions of the equations for steady compressible flow. Physical Rev. (2) **72**, 433 (1947).

A transformation of the equations for plane irrotational compressible fluid flow is outlined whereby the formation of solutions of these equations in closed form is reduced to the solution of a functional relationship connecting modified potential and stream functions. *E. N. Nilson.*

Hicks, B. L., Guenther, P. E., and Wasserman, R. H. New formulations of the equations for compressible flow. Quart. Appl. Math. **5**, 357–361 (1947).

The authors treat the problem of adiabatic flow of compressible fluids where the stagnation enthalpy along each stream line is constant, but different for different stream lines. In addition to the usual velocity vector **V**, two new languages are used: the Mach vector **M** which is a vector in the direction of **V** and equal in magnitude to the local Mach number, and the Crocco vector **W** which is a vector in the direction of **V** and equal in magnitude to the ratio of local velocity and the local limiting velocity. The theorem that the minimum stream tube section corresponds to the local Mach number unity is then proved. The authors then show that the fundamental fluid parameter in these flows is the stagnation pressure p_t, in the sense that for a given distribution of p_t, the same **W** or **M** (but not **V**) field is obtained. Some remarks on irrotational flows in **M**, **W** and **V** fields conclude the paper. *H. S. Tsien.*

Germain, Paul. Solution approchée des écoulements coniques infinitésimaux. C. R. Acad. Sci. Paris **225**, 487–489 (1947).

This is a continuation of the author's previous note [same C. R. **224**, 183–185 (1947); these Rev. **8**, 415]. If Z is the complex variable defining the conical field, let the real parts of $U(Z)$, $V(Z)$, $W(Z)$ give the actual disturbance velocities u, v, w. Let C_0 be the circle representing the Mach cone, with radius 1, and C the contour of the obstacle under consideration. Here C is given by the polar equation $\rho=\rho(\theta)$ and is within C_0; $\rho(\theta)$ and $\rho'(\theta)$ are bounded by a number $k\ll 1$. Then approximately $V(Z)=H(Z)-K_1Z$, where $K_1=2S/\pi\beta$, S is the area of C and $\beta^2=M^2-1$, M the Mach number; $H(Z)$ is regular outside C. If $z(Z)$ maps conformally the exterior of C on a circle γ of the z-plane, then let $F(Z)=i\log z/r$ with $z=re^{i\varphi}$ on the circle. Furthermore, let $G(Z)$ be a regular function outside C and real at infinity, with the real part on C equal to $2\beta^{-1}\rho^2d\theta/d\varphi$. Then $H(Z)=-iF'(Z)G(Z)$. This solves the problem. If two contours C, $Z_0(\theta)$ and $Z_1(\theta)$ are related by $Z_1=Z_0+\alpha$ (α being constant), then the corresponding H_0 and H_1 are related by $H_1=H_0+2\beta^{-1}\{\alpha r^2z^{-2}dz/dZ+\bar{\alpha}(1-dz/dZ)\}$. *H. S. Tsien* (Cambridge, Mass.).

Tukey, John W. Linearization of solutions in supersonic flow. Quart. Appl. Math. **5**, 361–365 (1947).

The author proposes to approximate the drag, lift, etc. of a body in supersonic flow by $Ap+Bq$, where A, B are constants, p is the free stream pressure and q is the dynamic pressure $\frac{1}{2}\gamma M^2p$, M free stream Mach number, γ ratio of specific heats. *H. S. Tsien* (Cambridge, Mass.).

★Tables of Supersonic Flow Around Yawing Cones, by the Staff of the Computing Section, Center of Analysis, Under the Direction of Zdeněk Kopal. Massachusetts Institute of Technology, Department of Electrical Engineering, Center of Analysis. Technical Report No. 3. Cambridge, Mass., 1947. xviii+323 pp. (2 plates).

[This continues "Tables of Supersonic Flow Around Cones," 1947; these Rev. **8**, 540.] Here the circular cone

is supposed to be fixed in a steady flow at a small angle ϵ of yaw. The equation of the cone, in spherical coordinates r, θ, ϕ, aligned with the flow, is simplified by neglecting terms of order ϵ^2 and smaller. The velocity components, pressure and density, are sought as perturbations of those for symmetrical flow past the same cone, again carrying terms proportional to ϵ. The flow behind the shock wave is assumed to be rotational, but frictionless and therefore isentropic along each streamline. The boundary conditions for this linearized system of equations are the condition at the cone and the Rankine-Hugoniot conditions at the shock wave. In this case the shock wave angle is $\theta=\theta_w+\epsilon\alpha\cos\phi$, where α is a constant determined by the solution. Thus, to this degree of accuracy, the shock wave is conical but is itself yawed through an angle $\alpha\epsilon$.

The problem is finally reduced to the integration of a second-order inhomogeneous ordinary differential equation whose coefficients are functions determined from the symmetrical-flow solutions. In part I, the results of 160 integrations are tabulated. Cone semi-apex angles θ_s range from 5° to 50°. The quantities tabulated permit calculation of the increments in velocity, pressure and density due to yaw, at all points between the cone and the shock wave. In parts II–V are tabulated and plotted various physical characteristics, such as surface velocities, shock-wave angle and position, and coefficients of integrated force components.

The author points out that the values of α in "second" solutions are many times as great as in "first" solutions; hence the "second" shock wave seems to be much more affected by yaw than the "first." This may have a bearing on the fact that only "first" positions seem to be observed in flight of projectiles. *W. R. Sears* (Ithaca, N. Y.).

✱Glauert, H. The Elements of Aerofoil and Airscrew Theory. 2d ed. Cambridge, at the University Press; New York, The Macmillan Company, 1947. iv+232 pp. $4.00.

The original edition of this book was published in 1926. Since then it has been a standard textbook on theoretical aerodynamics. The new edition has been revised in some minor respects by M. B. Glauert; for example, the non-dimensional coefficients of lift, drag and moment have been redefined in accordance with modern practice. Briefly, the content of the book is as follows: introduction to dynamics and kinematics of fluid flows, plane airfoil theory, Prandtl lifting-line wing theory, biplane theory and wind tunnel interference, elementary propeller theory. In this edition H. B. Squire has added, as an appendix, a set of notes, to which reference is made at various points in the text, indicating where present knowledge or practice deviates from the material in the text and giving references to sources of further information. A short bibliography of some well-known modern books on aerodynamics has also been appended. *W. R. Sears* (Ithaca, N. Y.).

Green, A. E. The two-dimensional aerofoil in a bounded stream. Quart. J. Math., Oxford Ser. **18**, 167–177 (1947).

A cylinder whose cross-section is given by the curve $\eta=0$, $-2\pi<\xi<0$, in the transformation

$$(*)\qquad z=e^{-i\zeta}\sum_{n=0}^{\infty}a_ne^{in\zeta},\qquad \zeta=\xi+i\eta,$$

is considered in plane flow in a uniform frictionless incompressible stream bounded by one plane wall at $y=-b$. The author begins with a suitable complex potential $w(z)$ that makes the wall a streamline, and evaluates the force $Y+iX$ and moment M in terms of its coefficients. The boundary condition at $\eta=0$ is then satisfied in series form; thus a series expression for Y is provided; $X=0$. The circulation is determined by the trailing-edge condition. The results are applied first to a thin circular-arc airfoil and then to a symmetrical airfoil of small thickness. In a closing section, the case of the general airfoil (*) in a stream bounded by two parallel walls is set up. *W. R. Sears.*

Nuzhin, S. G. Calculation of potential flow of an incompressible fluid past an airfoil of arbitrary shape. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **11**, 55–64 (1947). (Russian. English summary)

The approximate calculation of two-dimensional potential flows around wing profiles of arbitrary shape has been considered by various writers, for example, by von Mises, von Kármán, Lavrentieff, Theodorsen [Tech. Rep. Nat. Adv. Comm. Aeronaut., no. **452** (1933)] and Warschawski [Quart. Appl. Math. **3**, 12–28 (1945); these Rev. **6**, 207]. The author gives another approximate procedure for calculating the coefficients of the mapping function. Convergence is discussed and rapidity of convergence is illustrated in a special case by a numerical example. *D. C. Spencer.*

Laitone, E. V. The subsonic flow about a body of revolution. Quart. Appl. Math. **5**, 227–231 (1947).

The solution to the linearized equation for the subsonic irrotational flow about a body of revolution is expressed as an integral over an axial source distribution $f(x)$ by means of the Goldstein and Young linear perturbation theory of compressible flow [Ministry of Aircraft Production, Aeronaut. Res. Committee, Rep. and Memoranda no. **1909** (6865) (1943); these Rev. **6**, 193]. By means of a Taylor expansion the solution is obtained up to terms of first order in βr, where $\beta=(1-M^2)^{\frac{1}{2}}$ and r is the cylindrical coordinate. The boundary condition then determines $f(x)$ to a first order approximation: $f(x)=\beta U_\infty S'(x)$, where $S(x)$ is the cross-sectional area. All elements of the flow are thus given explicitly, to the first order of approximation, in terms of the Mach number M and the body shape. An expression is also given for the pressure coefficient to terms of the second order. It is difficult for the reviewer to judge, after the approximations involved, how accurate the final solutions really are. The results of several calculations using this method are shown. *D. Gilbarg* (Bloomington, Ind.).

Serebrisky, J. M. Flow past an aerofoil of arbitrary form. Engineering Rev. [Akad. Nauk SSSR. Inženernyĭ Sbornik] **3**, no. 1, 105–136 (1946). (Russian. English summary)

This paper presents a method for calculating the pressure distribution and field of velocities for an airfoil of arbitrary form in a two-dimensional flow of an ideal noncompressible fluid. The fundamental idea behind the method is the same as in Theodorsen's method [Tech. Rep. Nat. Adv. Comm. Aeronaut., no. **452** (1933)]. However, the method for carrying out the details of the iteration process is somewhat different. In addition, the author gives a method for obtaining approximately the flow about a given airfoil by starting with a particular airfoil from a given class of airfoils and adding corrections belonging to a particular class of functions in such a way as to obtain the original airfoil (this procedure actually takes place in Theodorsen's z'-plane). A number of tables and graphs are given to aid

in carrying out this procedure and in finding the resultant velocity distribution around the airfoil.

J. V. Wehausen (Falls Church, Va.).

Jones, W. Prichard. Theoretical determination of the pressure distribution on a finite wing in steady motion. Ministry of Supply [London], Aeronaut. Res. Council, Rep. and Memoranda No. **2145** (6711), 13 pp. (1943).

A method is suggested for calculating the pressure distribution on a deformable wing of any plan form in steady motion. It is based on vortex sheet theory, and is a modified form of Prandtl's acceleration potential method.

Extract from the paper.

Jones, Robert T. Properties of low-aspect-ratio pointed wings at speeds below and above the speed of sound. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **1032**, 12 pp. (5 plates) (1946).

Low-aspect-ratio wings having pointed plan forms are treated on the assumption that the flow potentials in planes at right angles to the long axis of the airfoils are similar to the corresponding two-dimensional potentials.

From the author's summary.

Brown, Clinton E. Theoretical lift and drag of thin triangular wings at supersonic speeds. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **1183**, 20 pp. (5 plates, 2 pp. errata) (1946).

Beginning with the elementary "source" solution of the linearized potential equation for supersonic small-perturbation flow, the author constructs the potential for a line of "doublets." A distribution of these over a triangular region satisfies the boundary conditions for a thin flat triangular wing at angle of attack. Lift and drag-due-to-lift are evaluated. For cases of subsonic leading edges (wing semi-apex angles less than the Mach angle) certain considerations regarding the suction at the leading edge in subsonic flow are carried over to evaluate the drag. All results agree with R. T. Jones's [see the preceding review] in the limiting case of very small apex angle. *W. R. Sears.*

Evvard, John C. The effects of yawing thin pointed wings at supersonic speeds. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **1429**, 17 pp. (9 plates) (1947).

Heaslet, Max A., Lomax, Harvard, and Jones, Arthur L. Volterra's solution of the wave equation as applied to three-dimensional supersonic airfoil problems. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **1412**, 56 pp. (21 plates) (1947).

Various aspects of the linearized theory of thin airfoils at supersonic speeds have been considered in a number of recent papers [cf. Puckett, J. Aeronaut. Sci. **13**, 475–484 (1946); Stewart, Quart. Appl. Math. (3) **4**, 246–254 (1946); these Rev. **8**, 109, and the three preceding reviews]. For the most part, the problems studied have been limited to those to which the Busemann theory of conical flows can be adapted. The purpose of the present paper is to develop a method of analysis which can be applied to much more general boundary conditions. In addition, it serves to put the usual physical arguments on a mathematical basis.

Volterra's solution of the two-dimensional wave equation is applied to the Prandtl-Glauert equation for linearized supersonic flow, $(1-M^2)\varphi_{xx}+\varphi_{yy}+\varphi_{zz}=0$, where M is the free-stream Mach number and φ is an acceleration potential. An application of Green's theorem for this equation yields solutions in terms of a surface integral. Integration over the portion of the wing contained within the upstream Mach cone of the point under consideration is sufficient for the problems considered.

Both the direct and inverse problems of lifting surface theory are treated, and the load distributions over flat plates with rectangular, trapezoidal and triangular plan forms are determined. Admittedly, the uniform treatment presented here is somewhat longer than the methods devised in the earlier papers for handling special cases; but the present development has the advantage of being applicable without modification to general boundary conditions.

E. N. Nilson (East Hartford, Conn.).

Heaslet, Max A. Compressible potential flow with circulation about a circular cylinder. Wartime Rep. Nat. Adv. Comm. Aeronaut., no. **A-42**, 24 pp. (6 plates) (1944).

The plane problem indicated in the title is attacked by the Rayleigh-Janzen theory, using essentially Poggi's technique, and computing terms of the order M^4, where M is the Mach number. The results include a series of graphs wherein the pressure coefficient at a point on the cylinder is plotted against M for several values of the circulation. For comparison, the same pressure coefficient is plotted according to the Glauert-Prandtl linear-perturbation theory, the von Kármán-Tsien approximation, and the Rayleigh-Janzen theory carrying only M^2 terms. [The same problem has been treated by Hasimoto and Sibaoka, Proc. Phys.-Math. Soc. Japan (3) **23**, 696–712 (1941); **25**, 575–577 (1943); these Rev. **8**, 542.] *W. R. Sears* (Ithaca, N. Y.).

Heaslet, Max. A. Compressible potential flow with circulation about a circular cylinder. Tech. Rep. Nat. Adv. Comm. Aeronaut., no. **780**, 9 pp. (1944).

Cf. the preceding review. *W. R. Sears* (Ithaca, N. Y.).

Reissner, Eric, and Stevens, John E. Effect of finite span on the airload distributions for oscillating wings. II. Methods of calculation and examples of application. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **1195**, 97 pp. (34 plates) (1947).

Lift and moment distributions are calculated for oscillating wings of finite span on the basis of the three-dimensional theory of part I [Reissner, same Notes, no. **1194** (1947); these Rev. **8**, 542]. The results obtained are compared with the corresponding results of the two-dimensional theory. *From the authors' summary.*

Falkovich, S. V., and Haskind, M. D. Vibration of a wing of finite span in a supersonic flow. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **11**, 371–376 (1947). (Russian. English summary)

[The authors' names appear in the opposite order at the beginning of the paper.] The case considered is that of a delta wing within a Mach cone.

From the authors' summary.

Gurevich, M. I. On the thin triangular wing at supersonic speed. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **11**, 395–396 (1947). (Russian. English summary)

This is an addendum to the paper of Falkovich and Haskind reviewed above.

Gurevich, M. I. Of supersonic flow about a triangular wing. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] 11, 297–300 (1947). (Russian. English summary)

Gurevich, M. I. Flow past an axi-symmetrical semi-body of finite drag. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] 11, 97–104 (1947). (Russian. English summary)

An axially-symmetric flow is defined by the velocity potential $\varphi(r,\theta)=A(r^nP_n(\cos\theta)-1)-Ur\cos\theta$, where r and θ are polar coordinates in a meridian plane and $-2\leqq n<1$. The corresponding stream function is

$$\psi(r,\theta)=-A(P_n\cos\theta-P_{n+1})r^{n+1}+\tfrac{1}{2}Ur^2\sin^2\theta.$$

The streamline for $\psi(r,\theta)=0$ is used to define an axially symmetric body for which $r=r_1$ when $\theta=0$ and $r\rightarrow\infty$ as $\theta\rightarrow\pi$ (except when $n=2$ which gives a sphere). The drag force on this class of bodies is then computed and the author finds that it is zero for $-2\leqq n<0$ and infinite for $0\leqq n<1$.

The author then attempts to find a body of finite, but positive, drag by using the velocity potential

$$\varphi(r,\theta)=\int_{-N}^{0}a(n)(r^nP_n-1)dn-Ur\cos\theta,\qquad -2\leqq-N\leqq0,$$

where it is assumed that (1) $\int_{-N}^{0}na(n)dn$ converges absolutely, and (2) for some sufficiently small $\epsilon>0$ the function $a(n)$ does not change sign in the interval $-\epsilon\leqq n\leqq0$. An axially-symmetric body may be defined by the streamline $\psi(r,\theta)=0$ for the stream function corresponding to $\varphi(r,\theta)$. Then, if D is the drag and x and y rectangular coordinates in a meridian plane, the author finds the asymptotic form of the body as $x\rightarrow-\infty$ to be

$$y\approx(8D/\rho\pi)^{\frac{1}{4}}(|x|^{\frac{1}{2}}/(\log|x|)^{\frac{1}{4}})(1-\log\log|x|/8\log|x|).$$

The asymptotic form of $a(n)$ as $n\rightarrow0$ is given by

$$a(n)\approx-(1/\pi)(D/-2\rho n^2)^{\frac{1}{2}}.$$

It is of interest to note that the asymptotic form of the body is identical with that obtained by N. Levinson for the asymptotic shape of an infinite cavity behind an axially symmetric body [Ann. of Math. (2) 47, 704–730 (1946); these Rev. 8, 419]. *J. V. Wehausen* (Falls Church, Va.).

Hantzsche, W., and Wendt, H. Conical tips in supersonic flow. Tech. Memos. Nat. Adv. Comm. Aeronaut., no. 1157, 13 pp. (14 plates) (1947).

[Translation of Jahrbuch 1942 der Deutschen Luftfahrtforschung, pp. 180–190.] In the case of cones in axially symmetric flow of supersonic velocity, adiabatic compression takes place between shock wave and surface of the cone. Interpolation curves between shock polars and the surface are therefore necessary for the complete understanding of this type of flow. They are given in the present report by graphical-numerical integration of the differential equation for all cone angles and airspeeds. *Author's summary.*

Weinig, F. Lift and drag of wings with small span. Tech. Memos. Nat. Adv. Comm. Aeronaut., no. 1151, 13 pp. (12 plates) (1947).

[Translation of Zentrale für Wissenschaftliches Berichtswesen der Luftfahrtforschung des Generalluftmeisters, Forschungsber. no. 1665 (1942).] The lift coefficient of a wing of small span at first shows a linear increase for increasing angle of attack, but to a lesser degree than was to be expected according to the theory of the lifting line; thereafter the lift coefficient increases more rapidly than linearly, as contrasted with the theory of the lifting line. The induced drag coefficient for a given lift coefficient, on the other hand, is obviously much smaller than it would be according to this theory. A small change in the theory of the lifting line will cover these deviations. *Author's summary.*

Ribner, Herbert S. The ring airfoil in nonaxial flow. J. Aeronaut. Sci. 14, 529–530 (1947).

Preston, J. H. The effect of the boundary layer and wake on the flow past a symmetrical aerofoil at zero incidence. I. The velocity distribution at the edge of, and outside the boundary layer and wake. Ministry of Supply [London], Aeronaut. Res. Council, Rep. and Memoranda no. 2107 (8796), 21 pp. (1945).

From considerations of the flux in various sections of the boundary layer the directions of the streamlines entering the boundary layer can be found. This gives an inner boundary condition for the potential flow outside the boundary layer. It is shown that this is equivalent to replacing the aerofoil by a new shape displaced out from the aerofoil by the amount of the "displacement thickness," and calculating the potential flow about it. This is done by the introduction of a doublet distribution of strength equal to the "displacement flux." Two alternative methods are given in the Appendix which lead to substantially the same result. In the first the vorticity at any section of the boundary layer or wake is concentrated at its C.G. in the form of a vortex sheet, and the velocity at any point outside the boundary layer can then be computed. This gives an answer almost identical with that obtained from the doublet distribution at the aerofoil surface.

In the second method, the velocity increments at any point outside the boundary layer, arising from vortex elements of known strength in the boundary layer and wake are summed. This method gives an answer which differs only slightly from the other two methods and is much longer. *From the author's summary.*

Fabri, Jean. Couche limite en écoulement supersonique. C. R. Acad. Sci. Paris 225, 42–44 (1947).

The author considers the case of a boundary layer in supersonic flow. The flow is divided into two regions, an outside flow of uniform velocity which depends only on the inclination of the dividing line and an inside flow in which the velocity drops to zero on the wall. The continuity equation is written in the form

$$\frac{d}{ds}[\rho U(\delta-\delta^*)]=\rho U\cos\theta\left(\frac{d\delta}{ds}-\tan\theta\right),$$

where δ^* denotes the displacement thickness of the boundary layer, θ the inclination of the outer velocity. It is assumed that δ and δ^* are expressible as power series in $\omega=\theta-\theta_0$, where θ_0 denotes an initial inclination of the velocity. For the case of the flat plate the following equation is obtained:

$$\frac{ds}{d\omega}=\frac{\omega^{r-1}A_0+A_1\omega}{(\theta_0+\omega)(1+C_1\omega)};$$

r, A_0, A_1 and C_1 are unknown constants. The dynamical equations are not considered. *H. W. Liepmann.*

Lees, Lester. The stability of the laminar boundary layer in a compressible fluid. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **1360**, 144 pp. (22 plates) (1947).

The paper is a continuation of an earlier investigation by L. Lees and C. C. Lin [same Tech. Notes, no. **1115** (1946); these Rev. **8**, 236] of the stability of laminar boundary layer flow in a compressible fluid. The method of small perturbations is used, as in the previous paper. The paper commences with the results of some calculations of the critical wave number at which a subsonic disturbance is undamped, for insulated surfaces at several values of free-stream Mach number M_0, and for conducting surfaces at $M_0=0.7$, for various ratios of surface temperature to free stream temperature. In each case a critical Reynolds number is found, below which all disturbances are damped.

Reverting to more general treatment, the author proves that a subsonic disturbance which is neutral in inviscid compressible fluid must be self-excited in a real fluid at sufficiently high Reynolds numbers. He also shows that, when the free stream velocity is subsonic, the boundary layer flow is unstable for sufficiently high Reynolds number, also that, whatever the value of M_0, the same will still be true if the product of mean density and mean vorticity is stationary for some value of $w>1-M_0^{-1}$ (where w is the ratio of the mean velocity component parallel to the surface to the free stream velocity). An approximate expression is developed for the limit of stability, the minimum critical Reynolds number, and this is used as a criterion in studying the influence of M_0 and thermal conditions on laminar stability. Heating the fluid through the solid boundary is shown to be destabilising, and cooling stabilising: these general trends are borne out by the numerical calculations presented at first.

For supersonic main stream velocities the above limit for w is positive and the laminar boundary layer may be stable at all Reynolds numbers or may be rendered stable for all Reynolds numbers by withdrawing heat through the solid boundary at a rate equal to or exceeding a certain critical value. Calculations for several values of M_0 from 1.3 to 5.0 show that for $M_0>3$ the critical rate of heat withdrawal through the surface is of the same order of magnitude as the rate of radiation from the surface in free flight conditions.

All this detailed work has indicated that the decisive factor in determining stability is the gradient of the product of density and vorticity. Assuming the same to remain true, it is conjectured that, when the free stream velocity is supersonic, laminar flow will be completely stable at all Reynolds numbers when there is a favourable pressure gradient above a certain critical value. *K. Mitchell.*

Liepmann, Hans Wolfgang, and Laufer, John. Investigations of free turbulent mixing. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **1257**, 38 pp. (21 plates) (1947).

The authors begin with a general survey of the problem of turbulence, indicating the unsatisfactory points of the existing theories. The laws of spread of turbulent jets, wakes, etc. are then discussed from energy and momentum relations, with the help of dimensional reasoning. It is shown by this method that they can be obtained without reference to a detailed theory of turbulence. Many useful experimental data for the two-dimensional mixing problem are included in this paper, from which several general conclusions are reached. It is found that neither the exchange coefficient nor the mixture length is constant across a section, although these assumptions lead to reasonable velocity distributions. The energy balance of the fluctuating motion is examined, and it is found that the rates of production, transport and dissipation are of the same order of magnitude. The maximum of these quantities is reached at approximately the middle of the mixing region. *C. C. Lin* (Cambridge, Mass.).

Cumming, Betty L. A review of turbulence theories. Austral. Counc. Aeronaut. Rep. **ACA-27**, 22 pp. (1946).

Chou, P. Y. The turbulent flow along a semi-infinite plate. Quart. Appl. Math. **5**, 346–353 (1947).

This paper is an application of the author's statistical theory of turbulence [see P. Y. Chou, same Quart. **3**, 38–54, 198–209 (1945); these Rev. **6**, 246; **7**, 346] to the turbulent flow along a semi-infinite plate. The results agree with the experimental data and the logarithmic law of von Kármán. The following three conditions are used. (1) The law of dynamical similarity is used in the sense that the mean quantities are functions of the dimensionless variable $\eta=yU_\tau/\nu$, where y is the distance from the plate, U_τ is the friction velocity and ν is the kinematic viscosity coefficient. (2) The quadruple correlations in the general theory are neglected. (3) The micro-scale of turbulence and the mean squares of the velocity fluctuations are assumed to be constant across the boundary layer. *C. C. Lin* (Cambridge, Mass.).

Huetz, Jacques. Calcul du couple exercé sur un ellipsoïde de révolution en rotation dans un milieu visqueux indéfini. C. R. Acad. Sci. Paris **225**, 40–42 (1947).

The author computes the moment exerted upon an ellipsoid of rotational symmetry by a viscous fluid. Stokes' approximation is used. The problem is of importance for certain types of viscosimeters. The final result for the moment M exerted by the fluid upon an ellipsoid of eccentricity e at an angle ω is

$$M=\frac{2}{3}\,\frac{8\pi\eta c^3\omega_0}{e/(1-e^2)-\log\{(1+e)/(1-e)\}^{\frac{1}{2}}};$$

c is defined by $r+is=c\cosh(\xi+i\eta)$, where r and s denote the original cylindrical coordinates; η is the dynamic viscosity coefficient. Experiments check the formula within 2%. *H. W. Liepmann* (Pasadena, Calif.).

Dolidze, D. E. Linear boundary problem for the unsteady motion of a viscous incompressible fluid. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **11**, 237–250 (1947). (Russian. English summary)

Les mouvements non permanents et lents d'un fluide visqueux incompressible sont, en l'absence des forces de masse, régis par le système linéaire de Stokes:

$$(1)\qquad \nu\Delta\mathbf{V}-\partial\mathbf{V}/\partial t=\rho^{-1}\operatorname{grad}p;\quad \operatorname{div}\mathbf{V}=0.$$

L'auteur se propose de construire le système **V**, p de solutions de (1) (la pression p n'étant définie qu'à une fonction additive du temps près), suffisamment régulière à l'intérieur (ou à l'extérieur) d'un domaine D (à deux ou à trois dimensions) telles que **V** se réduise (1) pour $t=0$, à une fonction des coordonnées, donnée à priori, dans le domaine occupé par le fluide et (2) sur la frontière F de ce domaine, à une fonction, donnée à priori, des coordonnées et du temps ($t\geqq 0$). Dans le cas du problème extérieur, l'allure de **V** à l'infini est assujettie à vérifier des conditions complémentaires dont la forme dépend du nombre des dimensions de D.

Après avoir établi un théorème d'unicité pour le problème fondamental ainsi posé, l'auteur utilise un procédé classique

en la matière et cherche l'inconnue **V** sous la forme $\mathbf{V}_0+\mathbf{V}^*$, avec $\mathbf{V}_0=\text{rot}\,\mathbf{W}$; **W** est alors solution du système:

$$(2)\qquad \nu\Delta\Delta\mathbf{V}-\partial\Delta\mathbf{V}/\partial t=0,$$

et se trouve déterminé pour $t=0$ en fonction des valeurs initiales de **V**. Il est aisé de construire une solution particulière de (2), se réduisant pour $t=0$ à une fonction connue des coordonnées. Compte tenu de la linéarité de (1), tout revient, des lors, à former la solution $\mathbf{V}^*$, p de (1), telle que **V** soit nul pour $t=0$ et se réduise sur F à une fonction connue à chaque instant. Pour aborder ce difficile problème, l'auteur étend d'abord aux systèmes du type (1) la méthode des solutions fondamentales d'Odqvist [cf. Math. Z. **32**, 329–375 (1930), ou H. Villat, "Leçons sur les Fluides Visqueux," Gauthier-Villars, Paris, 1943, pp. 268–286] et se ramène ainsi à un système d'équations intégrales de Volterra à noyaux singuliers, système qu'il résout au moyen d'un processus des approximations successives. L'auteur souligne l'intérêt qu'offrirait l'étude de la solution ainsi construite pour $t=\infty$. L'extrême concision des raisonnements de l'auteur rend pénible la lecture de son travail; on souhaiterait voir paraître un exposé plus détaillé de ses résultats, faisant, notamment, ressortir les hypothèses de régularité à faire sur les données et précisant divers points de ses démonstrations. Signalons, enfin, les omissions de la bibliographie de l'auteur en ce qui concerne les travaux fondamentaux d'Oseen [cf. Neuere Methoden und Ergebnisse in der Hydrodynamik, Leipzig, 1927, p. 67] et de Leray [Acta Math. **63**, 193–248 (1934)], qui établissent un théorème d'existence pour les régimes variables non linéaires, par des méthodes très différentes et sous des hypothèses de régularité tout autres. *J. Kravtchenko* (Grenoble).

Krzywoblocki, M. Z. On the two-dimensional steady flow of a compressible viscous fluid far behind a solid symmetrical body. J. Franklin Inst. **243**, 471–486 (1947).

The steady flow in the wake far behind a two-dimensional symmetrical body moving in a compressible viscous fluid is examined by a method of successive approximations closely akin to that used by Goldstein [Proc. Roy. Soc. London. Ser. A. **142**, 545–562 (1933)] for the incompressible case. The medium is assumed to obey the equation of state of a perfect gas and to have constant specific heats, while its viscosity and thermal conductivity may be functions of temperature. Approximation starts from the ordinary laminar boundary layer theory, with the additional assumption that the temperature gradient in the wake is negligible. Successful first approximations are obtained for the velocity distribution in the wake, and the second approximations are also successful, though expressed in terms of definite integrals which are not evaluated in the paper. An attempt to derive a third approximation breaks down, as in the incompressible case [Goldstein, op. cit.], the velocity correction failing to tend to zero at the wake boundary.

K. Mitchell (Newcastle-upon-Tyne).

Ursell, F. The effect of a fixed vertical barrier on surface waves in deep water. Proc. Cambridge Philos. Soc. **43**, 374–382 (1947).

Let the y-axis be taken vertically downwards and the barrier B extend from $(0, 0)$ to $(0, a)$ in the (x, y)-plane. The problem solved here consists in finding a function $\Phi(x, y, t)$ in the lower half-plane slit along B, which (a) satisfies the potential equation $\Phi_{xx}+\Phi_{yy}=0$; (b) is harmonic in the time: $\Phi_{tt}+n^2\Phi=0$; (c) satisfies the "free surface condition" $\Phi_{tt}+g\Phi_y=0$ for $y=0$; (d) satisfies the boundary condition $\Phi_x=0$ along B; (e) behaves like $Ae^{-Ky}\cos(Kx\pm nt+l)$ at infinity. The solution is obtained by superposition of exponential solutions which satisfy (a), (b), (c) only. The proper combination, which will satisfy (d) as well, is found by solving an integral equation. Explicit expressions for transmission and reflection coefficients in terms of Bessel functions are derived. The same method is applied to an infinite submerged vertical barrier extending from $(0, b)$ to $(0, \infty)$.

F. John (New York, N. Y.).

Taub, A. H. Refraction of plane shock waves. Physical Rev. (2) **72**, 51–60 (1947).

This paper is concerned with a two-dimensional problem in the reflection and refraction of shock waves. Two gaseous media, an "upper" and a "lower," are separated by a rectilinear interface. A shock wave in the upper medium is incident upon this interface. Five parameters are involved describing the constitution of the two gases and the strength and obliquity of the incident shock. The problem is to determine under what conditions a stationary three-shock configuration is possible, consisting of the incident shock and a reflected shock (both in the upper medium) and a refracted shock in the lower medium. The conditions to be satisfied are that (1) the flow upstream of the incident and refracted shocks be uniform, (2) the flow downstream of the reflected and refracted shocks be uniform, (3) the pressure be everywhere continuous across the interface, (4) the Rankine-Hugoniot equations hold across each shock. For specified gases and a given shock strength, the obliquity of the reflected shock is defined as an algebraic function of the obliquity of the incident shock by means of an equation of the twelfth degree. One branch of this function matches up in a certain sense with the acoustic approximation and this is assumed to be the physically realizable branch. The range of incident angles within which solutions of the assumed form can exist is then explored under each of two assumptions concerning the relative acoustical densities of the upper and lower media. The twelfth degree equation was solved numerically on the Eniac at the Ballistic Research Laboratory (Aberdeen, Md.) and a brief account of these computations is given. *D. P. Ling* (Murray Hill, N. J.).

Germain, Paul. Sur le problème de l'onde de choc détachée. C. R. Acad. Sci. Paris **224**, 1050–1052 (1947).

The author assumes a detached symmetric shock wave of known form in a steady two-dimensional flow. The oncoming flow is uniform, so that all pertinent data behind the shock are determined by the free flow Mach number. As coordinates of a point P downstream of the shock, he chooses s, the arc length P_0P along that streamline which joins P to a point P_0 on the shock, and α, the inclination of the shock at P_0. The dependent variables are q and τ, the magnitude and direction of the flow.

The entropy along each streamline is a known constant (adiabatic assumption) and ρ is a known function of q along each streamline (Bernoulli relation). For the rest, the equations of motion are two in number. By introducing two new auxiliary variables H and K, the author converts these to four quasi-linear equations in α and s, whose solution is determined by the initial data on the shock (in some limited region at least). Using these equations, he discusses the evaluation of the successive derivatives of ρ, τ, H and K at the point of symmetry of the shock wave. In particular, he

shows that all the derivatives with respect to s at this point of the product ρq can be evaluated directly from the equations, and he gives a brief discussion of their form.

D. P. Ling (Murray Hill, N. J.).

Hostinský, Bohuslav. Über Mittelwerte der Energie einer schwingenden Luftmasse, welche in einem Kasten eingeschlossen ist. Acad. Tchèque Sci. Bull. Int. Cl. Sci. Math. Nat. **43**, 24–36 (1942).

The author considers small irrotational vibrations of an air mass enclosed in a container. It is shown that the velocity potential has a space average independent of time and a time average (over an infinite interval) independent of position and equal to the space average. It is further shown that the time averages (over an infinite interval) of kinetic and potential energies are equal. The theorems are established by expanding the solutions of the wave equation for the velocity potential in eigenfunctions. *W. Kaplan.*

Sretensky, L. N. Theory of tides of long period. Bull. Acad. Sci. URSS. Sér. Géograph. Géophys. [Izvestia Akad. Nauk SSSR] **11**, 197–270 (1947). (Russian. English summary)

This is primarily an investigation of approximate formulae for the periods of tides on a rotating sphere entirely covered with water, and in polar seas. The author first obtains approximations by expansion in terms of $\beta=4a^2\omega^2/gh$, where a is the radius of the sphere, ω the angular velocity, and h the depth of the water. In the case of small values of β comparison with Hough's calculations [Philos. Trans. Roy. Soc. London, Ser. A. **189**, 201–257 (1897); **191**, 139–185 (1898)] and with those of Goldsborough [Proc. London Math. Soc. (2) **14**, 31–66 (1914); 207–229 (1915); Proc. Roy. Soc. London. Ser. A. **117**, 692–718 (1927)] and Goldsborough and Colborne [Proc. Roy. Soc. London. Ser. A. **126**, 1–15 (1929)] shows fair agreement. For large values of β the method adopted is that of asymptotic expansions of integrals of differential equations for large values of the parameter involved in these equations. This method is also applied to the determination of free oscillations of periods less than 12 hours. In the case of polar seas, the author claims that his method is exhaustive and compares well with the results found by Goldsborough by exact integration of the dynamical equations in the case of polar seas of angular radii 14° 30′ and 30°. *L. M. Milne-Thomson.*

Proudman, J. On the distribution of tides over a channel. Proc. London Math. Soc. (2) **49**, 211–226 (1946).

The channel is bounded by the planes $X=0$ and $X=1$. All transverse sections are equal, i.e., the depth is of the form $h=h(x)$. The motion is taken to be frictionless within the channel, although the effects of coastal friction are simulated in the boundary conditions. Let $\zeta(x, y)e^{i\sigma t}$ be a harmonic component of the tidal elevation due only to rotation of the earth. Then $\zeta(x, y)$ obeys the equation

$$\frac{\partial}{\partial x}\left(h\frac{\partial\zeta}{\partial x}\right)+h\frac{\partial^2\zeta}{\partial y^2}-\frac{i}{f}h'\frac{\partial\zeta}{\partial y}+\gamma\zeta=0,$$

where f, γ are constants depending on σ and the rotation of the earth. The boundary conditions are

$$\frac{\partial\zeta}{\partial x}-\frac{i}{f}\frac{\partial\zeta}{\partial y}=\frac{ik_0}{f}\zeta, \quad -\frac{ik_1}{f}\zeta$$

for $x=0, 1$, respectively, k_0, k_1 being constants so chosen that these conditions represent losses of energy at the boundaries equivalent to the effect of coastal friction. The solutions of the form $\zeta=Z(\lambda_s, x)e^{\lambda_s y}$ ($Z(\lambda_s, 0)=1$) are assumed to correspond to distinct values λ_s, and $Z_s(x)=Z(\lambda_s, x)$ are assumed analytic. These determine analogous functions $U_s(x)$, $V_s(x)$ for the transverse and longitudinal components u, v of the tide velocity. It is thus possible to express formal general solutions for ζ, u, v as expansions in terms of these basic solutions. Formal general expressions for ζ, u, v, which include the effect of external tide producing forces, are given when the tides due to these forces are known independently. The following expansion theorem is then proved. Let the values of the elevation and the longitudinal velocity on $y=0$ be functions $Z(x)$, $V(x)$, which are bounded and have bounded first derivatives in $[0, 1]$; then $Z(x)=\sum_s A_s Z_s(x)$ in $[0, 1]$ and $V(x)=\sum_s A_s V_s(x)$ in $(0, 1)$ for certain expansion coefficients A_s determined by both $Z(x)$ and $V(x)$, and the sets $\{Z_s(x)\}$, $\{V_s(x)\}$. However, the conditions for convergence of the formal solutions, e.g., $\zeta=\sum_s A_s Z_s(x)e^{\lambda_s y}$, for $y\neq0$, are left open. *D. Gilbarg* (Bloomington, Ind.).

Kalinin, N. K., and Polubarinova-Kochina, P. J. On unsteady motion of ground water with a free surface. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **11**, 231–236 (1947). (Russian. English summary)

Polubarinova-Kochina, P. J. Hydraulic theory of wells in stratified medium. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **11**, 357–362 (1947). (Russian. English summary)

Van Mieghem, Jacques. Interprétations énergétiques du critère d'instabilité de Kleinschmidt. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **31** (1945), 345–352 (1946).

The dynamic stability or instability of a geostrophic current can be studied by a consideration of the force acting on a particle of this current which has been displaced transversally to the current from its equilibrium position. If the force tends to bring the particle back to its former position the motion is stable, otherwise unstable. In the first case energy is given to the particle, in the second place energy is liberated by the displacement of the particle. The mathematical formulation of this consideration leads again to Kleinschmidt's criterion for dynamic stability if the displacement is isentropic. A consideration of the sum of kinetic and potential energies of the particle shows that in the stable case the potential energy acquired by the particle during its displacement along the inertia trajectory is positive and in the unstable case negative. *B. Haurwitz.*

Van Mieghem, Jacques. Les oscillations d'inertie du courant géostrophique. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **31** (1945), 547–555 (1946).

The mathematical analysis shows that two types of inertia oscillations may occur in a geostrophic current. The first type consists in an elliptic oscillation of the fluid particle transversal to the current on which a rectilinear oscillation of the same period in the direction of the current is superimposed. The second type consists of a hyperbolic oscillation transversal to the current on which a rectilinear oscillation in the direction of the current is superimposed.

B. Haurwitz (New York, N. Y.).

***Buchholz, Herbert. Die konfluente hypergeometrische Funktion bei der Berechnung des Schallfeldes einer punktförmigen Schallquelle im Zwischenraum zweier konfokaler Drehparabole und ihrer Entartungen.** Ber. Math.-Tagung Tübingen 1946, pp. 49–56 (1947).

The author reports briefly on his investigations of sound waves in the space between two confocal paraboloids of revolution. He uses paraboloidal coordinates: normal solutions of the wave equation in these coordinates contain trigonometric functions and confluent hypergeometric functions. The starting point is an expression of the simple spherical wave $R^{-1}e^{ikR}$ (R is the distance from an arbitrary fixed point) in terms of paraboloidal wave functions. A similar expression, with unknown coefficients, will represent the secondary field due to the presence of the paraboloidal walls, and appropriate boundary conditions serve to determine the unknown coefficients. The solution is represented by an infinite series of contour integrals. The evaluation of this solution is briefly discussed, and the author points out that limiting cases of his general solution embrace such physically different cases as the reflection of a spherical wave emitted from a point source inside a paraboloid of revolution, and the diffraction of a plane wave on a semi-infinite rod assumed to have the shape of a (thin) paraboloid of revolution. *A. Erdélyi* (Pasadena, Calif.).

Lax, M., and Feshbach, H. On the radiation problem at high frequencies. J. Acoust. Soc. Amer. 19, 682–690 (1947).

The authors are concerned with the acoustic radiation produced by a smooth cylinder of arbitrary cross-section (equation: $r=a(\varphi)$, in polar coordinates) when the surface pressure $S(\varphi)$ is prescribed (the problem of given velocity is analogous). The basic idea is to express the sound pressure as a superposition of plane waves travelling in complex directions,

$$p(r,\varphi)=\int D(u)\exp\{ikr\cos(\varphi-u)\}du.$$

The directionality factor $D(u)$ has to be determined from the boundary condition $p[a(\varphi),\varphi]=S(\varphi)$. This integral equation is approximately solved for high frequencies, that is, when the radius of curvature $\rho(\varphi)$ of the cross-section is large compared to the wave length. The methods involved, including saddle-point approximations, are suggested by physical principles. The factor $D(u)$ is split up into two factors, $g(u)$ and $\exp\{-ikb(u)\}$, where the former is slowly, the latter rapidly, varying with u. For convex cross-sections, the authors obtain $b(u)=a(h)\cos(h-u)$, where $h=h(u)$ is the function inverse to $u_0(\varphi)=\varphi-\tan^{-1}\{a'(\varphi)/a(\varphi)\}$. Further, in the first-order approximation,

$$S(\varphi)=\{ik\rho(\varphi)/2\pi\}^{-\frac{1}{2}}g\{u_0(\varphi)\},$$

which easily leads to the conventional geometric-optical approximation of the field of radiation.

A more thorough application of the method of steepest descents yields a numerically useful expansion in reciprocal powers of $k\rho$. The authors elaborate how abrupt changes in surface pressure produce typical diffraction effects. Nonconvex surfaces, under smooth-pressure conditions on the surface, are treated in an appendix. *C. J. Bouwkamp.*

Miles, John W. The diffraction of sound due to right-angled joints in rectangular tubes. J. Acoust. Soc. Amer. 19, 572–579 (1947).

Miles, John W. The equivalent circuit for a bifurcated cylindrical tube. J. Acoust. Soc. Amer. 19, 579–584 (1947).

Elasticity, Plasticity

Epstein, Paul S. On the elastic properties of lattices. Physical Rev. (2) 70, 915–922 (1946).

Die Cauchy-Relationen von W. Voigt zwischen den Elastizitätskoeffizienten eines Kristalles stehen oft in Widerspruch zu den Beobachtungen. Dies rührt von einem grundsätzlichen Mangel in den theoretischen Grundlagen dar. Die potentielle Energie eines deformierten Gitters kann in der Form $V=V_0+V_1+V_2$ (V_0=constant=Energie des undeformierten Gitters, V_1 linear in den Gitterpunktsverschiebungen aus der Normallage, V_2 quadratisch) geschrieben werden. Da die Normallage eine Gleichgewichtslage darstellt, wird $V_1=0$, sodass $V=V_2$ $(+V_0)$ ist. W. Voigt postulierte die Invarianz der Energie (V_2) gegenüber Drehungen und leitete darauf basierend die Cauchy-Relationen ab, übersah aber, dass V_2 die Energie nur unter der zusätzlichen Bedingung $V_1=0$, die gegenüber Drehungen nicht invariant ist, darstellt. Die Elastizitätskoeffizienten genügen daher den Cauchy-Relationen von W. Voigt nicht, sondern anderen erweiterten Gleichungen. Atomistisch gesehen sind die Abweichungen durch die Wechselwirkungen zwischen nicht unmittelbar benachbarten Teilchen (Koordination zweiter Sphäre) bedingt. *W. Nowacki* (Bern).

Shapiro, G. S. Les fonctions des tensions dans un système arbitraire de coordonnées curvilignes. C. R. (Doklady) Acad. Sci. URSS (N.S.) 55, 693–695 (1947).

The expressions for the displacements in the linear theory of elasticity and the components of the stress tensor are given in general curvilinear coordinates in terms of four biharmonic functions introduced by B. G. Galerkin [same C. R. (A) 1930, 353–358]. See also H. M. Westergaard [Bull. Amer. Math. Soc. 41, 695–699 (1935)] and R. D. Mindlin [ibid. 42, 373–376 (1936)]. *I. S. Sokolnikoff.*

***Moufang, R. Volumentreue Verzerrungen bei endlichen Formänderungen.** Ber. Math.-Tagung Tübingen 1946, pp. 109–110 (1947).

For infinitesimal strains the decomposition of the strain tensor into the strain deviation (determining the change of shape) and a spherical tensor (determining the change of volume) is well known. The author discusses the corresponding decomposition in the case of finite strains. *W. Prager* (Providence, R. I.).

***Signorini, Antonio. Recenti progressi della teoria delle trasformazioni termoelastiche finite.** Atti Convegno Mat. Roma 1942, pp. 153–168 (1945).

Moisil, Gr. C. Sur les petits mouvements des corps élastiques. Disquisit. Math. Phys. 1, 83–92 (1940).

Die Differentialgleichungen kleiner elastischer Bewegungen werden mit Hilfe hyperkomplexer Zahlen in ein

System von Differentialgleichungen erster Ordnung zerlegt. Für dieses wird die Möglichkeit der Integration diskutiert.
W. Nef (Fribourg).

Tagliacozzo, Carlo. Estensione del teorema del Menabrea ai mezzi elastici in vibrazione. Univ. Nac. Tucumán. Revista A. **5**, 153–168 (1946).

Tagliacozzo, Carlo. Sul moto elastico di un sistema nel suo stato naturale. Univ. Nac. Tucumán. Revista A. **5**, 169–186 (1946).

Tagliacozzo, Carlo. Un teorema di minimo sull'energia cinetica di un mezzo elastico in vibrazione libera. Univ. Nac. Tucumán. Revista A. **5**, 187–194 (1946).

These three papers deal with the motion of an elastic body which is subject to frictionless constraints. The mathematical formulation of the conditions of constraint is the same as that given by the author in an earlier paper [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (6) **23**, 432–436 (1936)], but the concept is not made entirely clear. It appears that some or all of the particles on the surface of the body are constrained to move on fixed surfaces, or on fixed curves, or are entirely fixed, and the surface stresses at the constraints are such that no work is done in an infinitesimal displacement satisfying the constraints. The papers are concerned chiefly with minimal principles, of which the following must serve as a sample. An elastic body subject to frictionless constraints moves under the action of body and surface forces which are assigned. Then the strain energy of the actual state of stress at any instant is less than the strain energy of any state of stress which differs from the actual state of stress by components which satisfy the equations of equilibrium under no body forces and which, on the surface, are of the type which may arise from the frictionless constraints.
J. L. Synge.

Prager, William. The general variational principle of the theory of structural stability. Quart. Appl. Math. **4**, 378–384 (1947).

The author first simplifies and extends earlier formulations of the characteristic-value problems associated with the general problem of structural stability, in both elastic and plastic ranges. Next an equivalent variational problem is formulated, leading to the possibility of further applications of direct methods in the calculus of variations in obtaining approximate solutions. The procedure is illustrated in a special case of lateral buckling of an elastic beam subject to thermal stresses.
F. B. Hildebrand.

Handelman, G. H., Lin, C. C., and Prager, W. On the mechanical behaviour of metals in the strain-hardening range. Quart. Appl. Math. **4**, 397–407 (1947).

This paper deals with stress-strain relations relevant to general states of stress and strain which can be reached by a single loading followed at most by one complete or partial unloading. It is shown that in the general case the assumption of the existence of a finite stress-strain relation for the first loading may lead to basic difficulties in the presence of neutral changes of stress, which correspond neither to loading nor to unloading, as there defined. Certain differential stress-strain relations are proposed and studied in some detail. These relations generalize previous results of Prager [Proc. Fifth Internat. Congr. Appl. Mech., Cambridge, Mass., 1938, pp. 234–237]. Methods of experimentally testing the validity of the several hypotheses are also suggested.
F. B. Hildebrand (Cambridge, Mass.).

Handelman, G. H., and Prager, W. Stress-strain relations for incompressible plastic materials with strain hardening. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **11**, 291–292 (1947). (English. Russian summary)

This note contains a brief presentation of the considerations involved in the paper reviewed above.
F. B. Hildebrand (Cambridge, Mass.).

Gleyzal, A. General stress-strain laws of elasticity and plasticity. J. Appl. Mech. **13**, A-261–A-264 (1946).

The stress-strain relation proposed in this paper is of the type which is called "theory of plastic deformation" in the recent Russian literature [see, for instance, A. A. Ilyushin, Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **9**, 207–218 (1945); these Rev. **7**, 144] and which Handelman, Lin and Prager called "finite stress-strain relation" in a recent paper [see the second preceding review]; it has the form of the generalized Hooke's law, but Young's modulus and Poisson's ratio are replaced by suitable invariants of the strain tensor.
W. Prager (Providence, R. I.).

Ilyushin, A. A. On the theory of plasticity in case of simple loading of plastic bodies with strain-hardening. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **11**, 293–296 (1947). (Russian. English summary)

The author discusses stress-strain relations for plastic materials with strain-hardening. He speaks of "simple" loading when the ratios between the various components of the stress deviation are maintained constant during the loading process. It is pointed out that the available empirical data concerning the mechanical behavior of plastic materials are almost exclusively concerned with simple loading and that the existing theories of plasticity furnish identical predictions when applied to simple loading. The author seems to consider this statement as an argument in favor of the simplest of these theories, the theory of small elastic-plastic deformations to which he has made many contributions. This manner of reasoning disregards the logical difficulties with which the theory of small elastic-plastic deformations is confronted in cases where partial unloading is involved [see, for instance, the third preceding review].
W. Prager (Providence, R. I.).

Markoff, A. A. Variation principles in the theory of plasticity. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **11**, 339–350 (1947). (Russian. English summary)

The author proves three minimum principles concerning the velocity field in an incompressible perfectly plastic solid which obeys the stress-strain relation of von Mises. The simplest of these principles is concerned with the case where the velocities are prescribed on the surface subject, of course, to the condition that the surface integral of the normal velocity component vanishes. If the velocity strain is denoted by e_{ij}, and if $E=(e_{ij}e_{ij})^{\frac{1}{2}}$, the minimum principle in question can be formulated as follows: among all velocity distributions which satisfy the boundary conditions and the equation of incompressibility, the actual distribution furnishes the smallest value for the volume integral of the invariant E. The author's other minimum principles are concerned with the cases where (a) the stresses are given on the surface or (b) the tangential stresses and the normal velocity. The minimum principles are used to establish uniqueness of the stresses in the interior (to within an arbitrary constant hydrostatic pressure). [The author assumes

tacitly that the boundary conditions assure plastic behavior throughout the body. In many practically important cases (e.g., indentation problems) this assumption is not justified.]
W. Prager (Providence, R. I.).

Oldroyd, J. G. Rectilinear plastic flow of a Bingham solid. II. Flow between confocal elliptic cylinders in relative motion. Proc. Cambridge Philos. Soc. 43, 521–532 (1947).

In an earlier paper [same Proc. 43, 396–405 (1947); these Rev. 8, 612] the author discussed the rectilinear flow of a Bingham solid between rigid circular cylinders, one of which is at rest while the other moves uniformly in the direction of its axis. In the present paper the discussion is extended to the case of confocal elliptic cylinders. *W. Prager.*

Scherman, D. J. Sur une méthode de résoudre certains problèmes de la théorie de l'élasticité pour les domaines doublement connexes. C. R. (Doklady) Acad. Sci. URSS (N.S.) 55, 697–700 (1947).

Let the elastic medium occupy a doubly connected domain S in the plane $z=x+iy$. The boundary L of S consists of two simple closed curves L_1 and L_2 having no points in common. Suppose L_1 is in the interior of L_2 and denote by S_1 and S_2 two simply connected domains bounded by L_1 and L_2, respectively. The domain S_1 is then infinite. The determination of stresses throughout the domain S, when forces are prescribed on L, can be reduced to the search for two functions $\varphi(z)$ and $\psi(z)$, regular in S, and satisfying on L the condition $\varphi(t)+\overline{t\varphi'(t)}+\psi(t)=f(t)$, where t is a variable point on L and $f(t)$ is a known function. The author indicates a method of reducing the solution of this problem to an analogous problem for a simply connected domain and is led to one integral equation of Fredholm type for a certain auxiliary function defined over one of the curves L_1 or L_2. *I. S. Sokolnikoff* (Los Angeles, Calif.).

Sherman, D. I. On the Dirichlet and Neuman problems in the theory of steady oscillations. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] 11, 259–266 (1947). (Russian. English summary)

For finite or infinite multiply-connected domains, the problems indicated in the title are reduced to Fredholm equation problems of such a sort that the existence of solutions can readily be established. *E. F. Beckenbach.*

Livens, G. H., and Morris, Rosa M. The boundary-value problems of plane stress. I. Philos. Mag. (7) 38, 153–179 (1947).

This is primarily an expository paper concerned with a systematic treatment of plane stress boundary value problems of elasticity by methods of the theory of functions of a complex variable. It correlates the work of several British investigators [W. G. Bickley, A. E. Green, S. Holgate, R. C. J. Howland, R. C. Knight, A. C. Stevenson and G. I. Taylor] on stress distribution in isotropic and aelotropic elastic plates containing holes. Although the authors cite some earlier work by M. Kolossoff and N. Muschelišvili, and refer to a short paper by S. Lehnickiĭ [Doklady Akad. Nauk SSSR (N.S.) 13 (1936 IV), 111–115 (1936)], a large amount of related work by Russian investigators escaped their attention. The reviewer believes that the bibliography contained in this paper would be more representative of the present status of two-dimensional problems of anisotropic elasticity if the following references were included: S. G. Lehnickiĭ, Appl. Math. Mech. [Acad. Nauk SSSR. Prikl. Mat. Mech.] 3, 56–69 (1936); N.S. 1, 77–90 (1937); 2, 181–210 (1938); 5, 71–91 (1941); D. Sherman, ibid. 9, 347–352 (1945); V. Morkovin, Quart. Appl. Math. 1, 116–129 (1943); 2, 300–352 (1945); these Rev. 7, 351; 5, 82; 6, 196].
I. S. Sokolnikoff (Los Angeles, Calif.).

Green, A. E. A concentrated force problem of plane strain or plane stress. J. Appl. Mech. 14, A-246 (1947).

The author gives a solution of the plane problem of the theory of elasticity for an infinite plate containing an elliptical hole. The boundary of the hole is loaded by a pair of concentrated forces at the ends of the minor axis of the ellipse; the solution appears in closed form in terms of the functions of a complex variable. A solution of the same problem was given in the form of an infinite series by P. S. Symonds [same J. 13, A-183–A-197 (1946); these Rev. 8, 358]. The procedure followed by the author is described in detail in N. Muschelišvili's book "On Certain Fundamental Problems of the Mathematical Theory of Elasticity," Moscow, 1935, pp. 260–287. *I. S. Sokolnikoff.*

Sadowsky, M. A., and Sternberg, E. Stress concentration around an ellipsoidal cavity in an infinite body under arbitrary plane stress perpendicular to the axis of revolution of cavity. J. Appl. Mech. 14, A-191–A-201 (1947).

An exact solution in closed form is given for the distribution of stress in the neighborhood of a cavity in the shape of an ellipsoid of revolution in an infinite elastic solid. The body is assumed to be in a state of plane stress perpendicular to the axis of revolution of the cavity. The authors extend the classical three-harmonic-function treatment of three-dimensional elastic problems of Boussinesq to orthogonal curvilinear coordinates and give a detailed discussion of stress concentration. *I. S. Sokolnikoff.*

Grioli, G. Sulle deformazioni elastiche dovute a due coppie in equilibrio. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) 5, 305–314 (1946).

Grioli, Giuseppe. Su di una semplice formula che lega le frequenze di una piastra anulare alla pressione applicata sul bordo. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) 6, 121–134 (1947).

The problem of the transverse vibrations of a circular ring plate is formulated in integral equation form. Numerical applications are included for the case of a plate clamped along the inner edge and simply supported along the outer edge. *E. Reissner* (Cambridge, Mass.).

Grioli, Giuseppe. Sul comportamento di un vincolo concentrato nel centro di una piastra circolare. Fac. Ci. Mat. Univ. Nac. Litoral. Publ. Inst. Mat. 8, 75–103 (1946). (Italian. Spanish summary)

Si discute la definizione di vincolo concentrato nel centro di una piastra circolare e la sua influenza sopra le vibrazioni sincrone e sopra il limite degli sforzi nel centro.
From the author's summary.

Ghizzetti, Aldo. Ricerche analitiche sul problema dell' equilibrio di una piastra indefinita a forma di striscia, incastrata lungo i due lati. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) 6, 145–187 (1947).

The deformation of an infinite strip perfectly clamped at its two parallel edges is ruled by the equations

$$(1)\quad (\text{div grad})^2\, u=F(x,y);\qquad u(0,y)=u(1,y)=u_x(0,y)=u_x(1,y)=0.$$

The author uses a Fourier transform with respect to y and

determines u under the assumptions that $F(x, y)$ satisfies a Hölder condition and that $F(x, y) \exp(-a|y|)$ is integrable in the infinite area of the strip, where a is the absolute value of the imaginary part of that root of $x+\sinh x=0$ which is not 0 and has the smallest absolute value. He proves that $u(x, y) \exp(-a|y|)$ tends uniformly to 0 as $|y|\rightarrow\infty$ and conjectures that if this result is added to (1), as a condition which u must satisfy, the solution of (1) becomes uniquely determined. The Green's function of the problem is thoroughly analyzed. *I. Opatowski.*

Volterra, E. Sul problema generale della piastra poggiata su suolo elastico. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **2**, 595–598 (1947).

Ishkova, A. G. An exact solution of the problem of bending of a round plate supported by an elastic half-space and subjected to an axially-symmetrical uniformly distributed load. C. R. (Doklady) Acad. Sci. URSS (N.S.) **56**, 129–132 (1947).

The problem stated in the title is reduced to that of determining the contact pressure p between the plate and the half-space. This function is taken in the form $p=A(1-\rho^2)^{-\frac{1}{2}}+\sum_{n=0}^{\infty}b_{2n}\rho^{2n}$, where ρ is the radial cylindrical coordinate. Explicit expressions are found in the form of triple infinite series for the b_{2n} and for A. The author states that she has established the convergence of the development for p. She thus concludes that the "guessed" singularity term $(1-\rho^2)^{\frac{1}{2}}$ is rigorously the irregular part of the solution. *G. F. Carrier* (Providence, R. I.).

Conway, H. D. Large deflections of circular and square plates. Philos. Mag. (7) **37**, 756–767 (1946).

The principle of virtual displacement is used to calculate approximate deflections and stresses in uniformly loaded and circular square plates so supported at the edges as to permit rotation on loading. The procedure is that outlined in A. and L. Föppl's "Drang und Zwang," vol. 1 [Munich, 1920]. *I. S. Sokolnikoff* (Los Angeles, Calif.).

Conway, H. D. The large deflections of rectangular membranes and plates. Philos. Mag. (7) **37**, 767–778 (1946).

The calculations of the paper reviewed above are extended to rectangular plates and membranes having length-breadth ratio ranging from 1 to 4. *I. S. Sokolnikoff.*

Smith, R. C. T. The buckling of plywood plates in shear. Austral. Counc. Aeronaut, Rep. **ACA-29**, 24 pp. (1946).

Truesdell, C. On Sokolovsky's "momentless shells." Trans. Amer. Math. Soc. **61**, 128–133 (1947).

The author points out that Sokolovsky [Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **1**, 291–306 (1937)] had obtained results on momentless shells which were also obtained in more general form by P. Nemenyi and the author [Proc. Nat. Acad. Sci. U. S. A. **29**, 159–162 (1943); these Rev. **5**, 84] as well as others not obtained by them. It is shown that some of the solutions obtained by Sokolovsky can be still further generalized. *J. J. Stoker* (New York, N. Y.).

Goldenweiser, A. L. Momentless theory of shells whose middle surface is of a curve of the second order. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **11**, 285–290 (1947). (Russian. English summary)

[A more accurate translation of the Russian title would omit the phrase "of a curve."] This note indicates that the solution of the problems of momentless shell theory for shells whose middle surfaces are quadrics is reducible, by a suitable change of dependent and independent variables, to the integration of Poisson's or wave equations. *I. S. Sokolnikoff* (Los Angeles, Calif.).

Hadji-Argyris, J., and Dunne, P. C. The general theory of cylindrical and conical tubes under torsion and bending loads. Single and many cell tubes of arbitrary cross-section with rigid diaphragms. II. J. Roy. Aeronaut. Soc. **51**, 757–784 (1947).

[Part I appeared in the same vol., 199–269 (1947); these Rev. **8**, 613.] The present part continues the developments of part I. *E. Reissner* (Cambridge, Mass.).

Klinger, Friedrich. Die Statik und Kinematik des räumlich gekrümmten, elastischen Stabes. Die vollständige Integration des Systems der Differentialgleichungen der elastischen Linie eines beliebig geformten, nach einer beliebigen räumlichen Kurve verlaufenden, an den Enden beliebig gelagerten, beliebigen Belastungen, Temperatur- und Schwindwirkungen unterworfenen, gekrümmten, elastischen Stabes. Akad. Wiss. Wien, S.-B. IIa. **151**, 13–80 (1942).

The author's summary is as follows. Die vorliegende Arbeit behandelt die Formänderungstheorie erster Ordnung des gekrümmten Stabes. Die Losung des gestellten Problems gelingt unter durchgreifender Verwendung der Vektor- und Dyadenrechnung sowie des Matrizenkalkuls. In der Baustatik werden die Begriffe Kraft und Moment, Verdrehung und Verschiebung meist nur gesondert ins Auge gefasst. Abweichend hiervon werden in der vorliegenden Untersuchung die gegebenen Einzelbegriffe zu Dynamen und Schrauben zusammengefasst. Es wird gleich mit den Dynamen und Schrauben selbst gerechnet. Der vorliegende Teil der Schrift behandelt die Theorie der räumlich gekrümmten, elastischen Stäbe. In einer folgenden Untersuchung sollen die hier entwickelten Formeln für besondere Lagerungsarten des räumlich gekrümmten Stabes sowie auch der ebenen Bogentrager und der Träger mit gerader Stabachse spezialisiert werden. Weiters soll eine Anwendung auf die ebenen Rahmenkonstruktionen mit Berücksichtigung der Verdrehbarkeit und der Verschieblichkeit der starren Stabecken gebracht werden. *F. B. Hildebrand* (Cambridge, Mass.).

Tranter, C. J., and Craggs, J. W. Stresses near the end of a long cylindrical shaft under non-uniform pressure loading. Philos. Mag. (7) **38**, 214–225 (1947).

Kempner, Joseph. Recurrence formulas and differential equations for stress analysis of cambered box beams. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **1466**, 50 pp. (1947).

Stagni, Ernesto. Un metodo numerico per la determinazione del carico critico nelle strutture lineari. Ann. Mat. Pura Appl. (4) **24**, 237–256 (1945).

Cattaneo, Carlo. Pressione eccentrica di un cilindro rigido a base ellittica sopra un suolo elastico. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **6**, 203–214 (1947).

Galin, L. A. Indentation of a punch of elliptic shape in plane in an elastic semi-space. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **11**, 281–284 (1947). (Russian. English summary)

Haag, Jules. Théorie des fils élastiques. J. Math. Pures Appl. (9) **25**, 1–92 (1946).

The author considers the elastic deformation of a wire with a variable cross section such that all cross sections in the deformed wire are similar to a certain plane region, the proportionality factor being a quantity t which varies along the wire. Both body and surface forces act on the wire. Large displacements are permitted, provided the deformation is elastic.

Eulerian coordinates (x, y, s) are employed, where x and y are rectangular Cartesian coordinates relative to the principal axes of inertia of the cross section of the deformed wire, and s is the arc length of the neutral axis of the deformed wire. The microscopic equations of equilibrium (involving the stress components) and the macroscopic equations of equilibrium (involving the equilibrium of the portion of the wire between adjacent cross sections) are determined in terms of this coordinate system. This determination is carried out without the use of the tools of the tensor calculus, and hence is rather involved. Displacement functions, stress-strain relations and the boundary conditions are also introduced. Solutions are sought for this set of equations in the form of power series in t. This reduces the integration of this set of equations to an iteration process which yields approximate solutions. Observations are made concerning the order of magnitude of the external loads consistent with small strain. Two iterations are actually carried out. Several applications of the theory are presented, among which are applications to the wire with a circular cross section and to the straight wire. *G. E. Hay* (Ann Arbor, Mich.).

Pinney, Edmund. Vibration modes of tapered beams. Amer. Math. Monthly **54**, 391–394 (1947).

The eigenvalue problem associated with vibrating beams is characterized by the differential equation

$$[\alpha(x)y''(x)]''-\lambda\rho(x)y(x)=0$$

and certain homogeneous boundary conditions. In this paper explicit solutions in the form of Bessel functions are given which correspond to a limited set of $\alpha(x)$, $\rho(x)$. For slowly varying α, ρ, a perturbation technique is proposed which, however, leads to no possibility of improving the first estimate for the eigenvalue. *G. F. Carrier.*

Riz, P. M. Large oscillations of a string under arbitrary initial stretching. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **11**, 389–390 (1947). (Russian. English summary)

One particularity of the oscillations of a string under large strains is pointed out: the possibility of the nonlinear system of equations of the oscillations passing into linear equations. The first possibility (for small oscillations) is well known. The second, which may be of theoretical interest, is the case of the equality of the rigidity of the string and its initial stretching, when the coefficient γ is equal to zero. *Author's summary.*

Rakhmatulin, K. A. Impact on a flexible cord. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **11**, 379–382 (1947). (Russian. English summary)

Continuation of a paper in the same journal **9**, 449–462 (1945); these Rev. **7**, 351.

Cooper, J. L. B. The propagation of elastic waves in a rod. Philos. Mag. (7) **38**, 1–22 (1947).

Two-dimensional waves in an elastic plate are considered. The admissible displacements are u_x, u_y, where the (x, z)-plane is the neutral surface of the plate. Using the exact equations of elasticity, and a Fourier transform integrating technique, conclusions are obtained as to (1) the velocities of propagation which can be obtained and in particular their upper bounds; (2) the dispersive nature of the waves, both longitudinal and transverse; (3) the velocity at which elastic energy can be expected to be transported. *G. F. Carrier* (Providence, R. I.).

Zvolinsky, N. V. Plane waves in an elastic semi-space covered with a liquid layer. C. R. (Doklady) Acad. Sci. URSS (N.S.) **56**, 19–22 (1947).

Scholte, J. G. The range of existence of Rayleigh and Stoneley waves. Monthly Notices Roy. Astr. Soc. Geophys. Suppl. **5**, 120–126 (1947).

The author, referring back to his previous papers on the subject [Nederl. Akad. Wetensch., Proc. **45**, 20–25, 159–164, 380–386, 449–456, 516–523 (1942); these Rev. **5**, 252], considers the problem of the range of existence of the Stoneley type of elastic surface waves in a layered medium in the two limiting cases, when the wave length of curl waves in the medium is very small and when it is very large in comparison with the thickness of the layer. A general algebraic treatment of the Stoneley wave equation is said to show that Stoneley waves can exist only in limited ranges of rigidity and density ratios in the two media. The author gives limiting curves for the two cases plotted with the rigidity ratios as ordinates and the density ratios as abscissae. Equations of the boundary curves are given. The range of existence of the generalized Rayleigh and Stoneley wave systems is determined and a number of particular cases are evaluated. *J. B. Macelwane* (St. Louis, Mo.).

MATHEMATICAL PHYSICS

Optics, Electromagnetic Theory

Simon, J. Contribution à l'étude des aberrations du 3ᵉ ordre d'un système centré. Rev. Optique **26**, 121–144 (1947).

By considering the wave-fronts emerging from a centered optical system as elliptical paraboloids (which in certain cases become paraboloids of revolution) the characteristics of third-order aberrations can be obtained. The relation is shown between the aberrations on the one hand and the geometrical characteristics of the paraboloids and derived surfaces on the other. *A. J. Kavanagh* (Buffalo, N. Y.).

Conn, G. K. T. Optical theory of the echelette grating. Proc. Cambridge Philos. Soc. **43**, 240–253 (1947).

An elementary treatment is given of the optical theory of reflecting echelette gratings, used in the infra-red. Attention is directed mainly to the case of fixed angle between incident beam and the diffracted beam under consideration. Expressions are given for the relation between wave-length and angle of rotation of the grating, the dispersion, the resolving power and the intensity distribution. The effect on the intensity distribution of modifiers such as light-source and fore-prism characteristics is discussed. The use

of high orders of interference to permit utilization of coarse gratings with satisfactory resolution is considered.

A. J. Kavanagh (Buffalo, N. Y.).

***Cosslett, V. E. Introduction to Electron Optics. The Production, Propagation and Focusing of Electron Beams.** Oxford, at the Clarendon Press, 1946. viii+272 pp. (8 plates). $6.50.

The first chapter contains a short account of the physical principles which lead to the ideas of electron optics. Here one finds also a brief treatment of focusing electrons in constant electric and magnetic fields. In the second chapter the mathematical analysis of the electrostatic field is presented in a concise form, especially for axially symmetric fields. A brief discussion of the relaxation method for solving Laplace's equation in two dimensions and a description of the electrolytic trough are also included. The author concludes the chapter by including a brief analysis of the motion of electrons in axial fields and ray tracing methods. Chapter three contains a discussion of different types of electrostatic lenses, their geometrical properties and the electron mirror. In chapter four the important magnetic lenses are analyzed in some detail, including a short analytical treatment of Glaser's magnetic lens [cf. Glaser and Lammel, Arch. Elektrotechnik **37**, 347–356 (1943); these Rev. **7**, 398]. There follows a clear analysis of combined lenses in which the author follows Brüche and Scherzer [Geometrische Elektronenoptik, Springer, Berlin, 1934]. In chapter five aberrations are treated mathematically following Scherzer [Z. Physik **101**, 593–603 (1936)] and Voit [Z. Instrumentenkunde **59**, 71–82 (1939)]. One also finds a clear analysis of various image errors, including chromatic aberration and the effect of space charge on the image formation.

In the following two chapters a brief but good analysis of the principles and methods of producing electrons from metals is presented. The cathode ray tube and other tubes, such as television tubes, are discussed at some length. In chapter eight there is a short account of diffraction of electrons and its application to the electron microscope. In the following chapter the author gives a clear exposition of various applications of electron optics in the construction of modern apparatus and instruments such as the beta-ray spectrometer, the cyclotron, the magnetron, the betatron, etc. The last chapter contains a short account of velocity modulated tubes such as the clystron and other high frequency generating valves.

There is one appendix in which the author presents an analytic treatment of electron optics including the theory of aberrations by the Hamilton method of characteristics. It follows closely the treatment as presented by Picht [Einführung in der Theorie der Elektronenoptik, Barth, Leipzig, 1939]. Finally, a short bibliography, tables of important physical constants and relations, a lengthy symbol explanation list and an author-subject index cover the final pages of the book, which is profusely illustrated with excellent diagrams and photographs of electron optical instruments.

N. Chako (Auburn, Ala.).

Cabannes, Jean. Les théories de la lumière. Rev. Optique **26**, 333–353 (1947).

Expository lecture.

Weiss, P. Applications of Kelvin's transformation in electricity, magnetism and hydrodynamics. Philos. Mag. (7) **38**, 200–214 (1947).

It is shown that the application of Kelvin's inversion theorem [see for example, O. D. Kellogg, Foundations of Potential Theory, Springer, Berlin, 1929, p. 232] gives a compact expression for the solution of electrostatic, magnetostatic, and hydrodynamic problems. These problems lead to the integral

$$g_{n,k}(x)=nx\int_0^1 (1-sx)^{-(n+1)}s^k ds.$$

A rapidly converging expression for this integral in the neighborhood of $x=1$ is developed.

C. Kikuchi.

Humblet, J. Expression approchée de la composante normale du tenseur de Maxwell en fonction du vecteur de Poynting. Physica **13**, 17–20 (1947).

It is shown that the normal component of the Maxwell tensor, defined and designated by

$$T\mathbf{R}^0=\frac{1}{4\pi}[\mathbf{E}.\mathbf{E}\times\mathbf{R}^0+\mathbf{H}.\mathbf{H}\times\mathbf{R}^0-\tfrac{1}{2}(E^2+H^2)\mathbf{R}^0]$$

is equal to the negative of the Poynting vector divided by c, for points sufficiently removed from sources. [Note the difference in notation. The author uses the symbols $\times$ and $\wedge$ for the dot and cross products of vectors, respectively. The symbol . is used to denote ordinary scalar multiplication.]

C. Kikuchi (East Lansing, Mich.).

Podolsky, Boris. On the Lorentz transformation of charge and current densities. Physical Rev. (2) **72**, 624–626 (1947).

It is shown that the charge-current density vector $S_\alpha=(j_x, j_y, j_z, i\rho)$, where

$$\mathbf{j}=\sum_s (e_s/c)\mathbf{u}_s(t)\delta(\mathbf{r}-\mathbf{r}_s(t)),$$

$$\rho=\sum_s e_s\delta(\mathbf{r}-\mathbf{r}_s(t)),$$

is invariant under Lorentz transformation.

C. Kikuchi.

Bloch, Léon. Sur l'énergie électromagnétique d'un système isolé. Rev. Gén. Électricité **56**, 270–275 (1947).

The electromagnetic energy of a field containing electric and magnetic dipoles, as well as point charges, is investigated. This expression reduces to a very simple form for a static system. It is pointed out that a similar expression for the energy is obtained by a parallel calculation for the meson field.

C. Kikuchi (East Lansing, Mich.).

Meixner, Josef. Das Babinetsche Prinzip der Optik. Z. Naturforschung **1**, 496–498 (1946).

The paper deals with the rigorous form of Babinet's principle in the theory of electromagnetic diffraction [von Laue, Handbuch der Experimentalphysik, v. 18, Leipzig, 1928, p. 344]. Referring to Debye [Ann. Physik (4) **30**, 57–136 (1909)], the author states that the electromagnetic field is derivable from two scalar wave potentials. The two mutually complementary problems of diffraction are solved in terms of these potentials, by the reviewer's method in his thesis relating to acoustic diffraction [University of Groningen, 1941; cf. these Rev. **8**, 179].

In the reviewer's opinion, the author's proof of the rigorous principle of Babinet relating to plane electromagnetic diffraction problems is not exhaustive; see, in this connection, a note by the reviewer [Physica **12**, 467–474 (1946); these Rev. **8**, 363]. Moreover, the author's formulation of the principle seems to be inconsistent with the result of Copson [Proc. Roy. Soc. London. Ser. A. **186**, 100–118

(1946); these Rev. **8**, 179]. In Copson's notation, the principle here amounts to

$$-\mathbf{E}^1+\mathbf{H}^2=\mathbf{F}, \quad -\mathbf{H}^1-\mathbf{E}^2=\mathbf{G},$$

which is incorrect with respect to the algebraic signs of $\mathbf{E}^1$, $\mathbf{H}^1$. Copson's formulation is compatible with Sommerfeld's solution for the half-plane. *C. J. Bouwkamp.*

Arzeliès, Henri. La réflexion vitreuse. Ann. Physique (2) 1, 5–69 (1946).

The author discusses, on the basis of Maxwell's equations, the theory of reflection and refraction of plane electromagnetic waves at a plane surface between two nonabsorbing media of given dielectric constants K_1, K_2 and magnetic permeabilities μ_1, μ_2. The discussion covers the evanescent waves associated with total internal reflection and the results are applied to investigate the transmission of plane waves through thin plates, including the case of parallel light incident on a thin plane lamina, of refractive index less than those of the surrounding media, at an angle exceeding the critical angle. *E. H. Linfoot* (Bristol).

Arzeliès, Henri. Réflexion sélective et réflexion métallique. Ann. Physique (2) 2, 133–194 (1947).

In this paper, which is a sequel to the paper reviewed above, the author considers the reflection and refraction of plane electromagnetic waves at a plane surface separating media which may be conducting or absorbing. The investigation is based on Maxwell's equations and the properties of the media are assumed to be due to the presence of free and bound electrons and ions, which are set into oscillation by the waves. The results are applied to discuss reflection and transmission by an absorbing plane lamina and by an absorbing prism. *E. H. Linfoot* (Bristol).

Arzeliès, Henri. Sur l'énergie réfléchie par une lame absorbante. C. R. Acad. Sci. Paris **225**, 184 (1947).

Abelès, Florin. Sur la propagation normale des ondes dans un milieu stratifié non magnétique. C. R. Acad. Sci. Paris **225**, 569–571 (1947).

Perrot, Marcel. Étude de quelques propriétés optiques des lames "antireflets." Rev. Optique **26**, 295–306 (1947).

***Watson, W. H. The Physical Principles of Wave Guide Transmission and Antenna Systems.** Oxford, at the Clarendon Press, 1947. xiv+208 pp. (3 plates). $7.00.

This text is a survey of some of the theoretical methods as well as some of the apparatus which has been developed in microwave theory in recent years. From the mathematical viewpoint the main interest in this book is chapter X, in which a partial study of the integration methods of the Maxwell equations is made. We find here an indication of the application of the theory of integral equations to these boundary value problems, as well as some indication of the question of field representation. [In view of the fact that this book was in type before the publication of the reports of some British and American organizations, this chapter is necessarily sketchy.] The remaining chapters discuss the theory of wave guides and the theory of the various types of obstacles which are used in practice. In some cases a fairly good theoretical picture is available, in other cases not. *A. Heins* (Pittsburgh, Pa.).

Infeld, L. The influence of the width of the gap upon the theory of antennas. Quart. Appl. Math. **5**, 113–132 (1947).

The gap in a spherical antenna is introduced by considering a field E_θ' applied at the surface of a metallic sphere of radius a. If we write $E_\theta'=-Vf(\theta)/a$, where $\int_0^\pi f(\theta)d\theta=1$, the applied voltage is $V=-a\int_0^\pi E_\theta' d\theta$. In the theory developed by Stratton and Chu [J. Appl. Phys. **12**, 230–248 (1941)] a Dirac function $f(\theta)=\delta(\frac{1}{2}\pi-\theta)$ is assumed, but this leads to a divergent series for the imaginary part of the input admittance of the antenna at the equatorial plane, $\theta=\frac{1}{2}\pi$. The author introduces a function

$$f(\theta)=2^{-2s-1}\frac{(2s+1)!}{s!s!}\sin^{2s+1}\theta,$$

where s is an integer, and thence obtains for the input admittance the finite series

$$Y(\theta)=2\pi\sin\theta \times\sum_{m=0}^{s}\frac{(-1)^m(s+1)(2s+1)!(s+m+1)!2m!(4m+3)}{(2s+2m+3)!(s-m)!m!(m+1)!Z_{2m+1}} \times\dot{P}_{2m+1}(\cos\theta).$$

As $s\to\infty$ and $\theta\to\frac{1}{2}\pi$ this reduces to the Stratton-Chu formula, but for a small finite gap the value of Y at the edge of the gap $(\theta=\frac{1}{2}\pi\pm\frac{1}{2}d/a)$ is

$$Y\left(\frac{d}{2a}\right)= 2\pi\cos\frac{d}{2a}\sum_{m=0}^{M}\frac{(-1)^m(2m)!(4m+3)}{2^{2m+2}(m+1)!m!Z_{2m+1}}\dot{P}_{2m+1}\left(\sin\frac{d}{2a}\right) -\frac{iak}{60\pi}\operatorname{Ci}\left(\frac{Md}{a}\right).$$

The final term represents the sum of the terms of the series for $m>M$ (an arbitrarily chosen large number), and shows the nature of the singularity of the admittance when $d\to 0$. A similar correction term is derived for the spheroidal antenna. *M. C. Gray* (New York, N. Y.).

Magnus, Wilhelm. Zur Theorie des zylindrisch-parabolischen Spiegels. Z. Physik **118**, 343–356 (1941).

The focal line of a perfectly conducting reflector in the shape of a parabolic cylinder is a source of electromagnetic waves. The focal line is chosen as the z-axis, the primary electric field is parallel to the focal line and is given by $e^{i\omega t}H_0^{(2)}(k(x^2+y^2)^{\frac{1}{2}})$. In order to find the reflected field, the author transforms to the coordinates ξ, η of the parabolic cylinder ($kx=\xi\eta$, $2ky=\xi^2-\eta^2$). He represents the incident field as

$$H_0^{(2)}(\tfrac{1}{2}(\xi^2+\eta^2))= \frac{e^{-i(\xi^2+\eta^2)}}{\pi^2\sqrt{2}}\int_{\nu-i\infty}^{\nu+i\infty}\delta(\xi,n)\delta(\eta,-n-1)\Gamma(-\tfrac{1}{2}n)\Gamma(\tfrac{1}{2}(n+1))dn,$$

$-1<\nu<0$, where $\delta(\xi,n)=e^{\frac{1}{2}i\xi^2}D_n((1+i)\xi)$, and the reflected field in the form of a similar integral, obtains a series expansion of the total field by evaluating the contour integral by the calculus of residues, and gives an expression for the radiation resistance. A mathematical appendix gives the derivation of some of the formulae used and includes also a representation of a spherical wave in terms of wave functions of the parabolic cylinder. *A. Erdélyi.*

Kinzer, J. P., and Wilson, I. G. Some results on cylindrical cavity resonators. Bell System Tech. J. **26**, 410–445 (1 plate) (1947).

The paper presents in condensed form all quantitative data that are useful in the engineering practice of cavity resonators (rectangular prisms, circular cylinders, full coaxials). The mathematics involved is mainly elementary and of an approximate character. Some data concerning elliptic-cylinder cavities (Mathieu functions) are stated. The paper concludes with an extended bibliography of 89 items, relating also to wave guide and acoustical problems.

The reviewer would like to add a reference to work that seems to have generally escaped attention: R. Weyrich, J. Reine Angew. Math. **172**, 133–150 (1934).

C. J. Bouwkamp (Eindhoven).

Jouguet, Marc. Propagation dans les tuyaux courbés. C. R. Acad. Sci. Paris **224**, 107–109 (1947).

Jouguet, Marc. Sur la propagation dans les guides courbés à section circulaire. C. R. Acad. Sci. Paris **224**, 549–551 (1947).

Pekeris, C. L. The field of a microwave dipole antenna in the vicinity of the horizon. J. Appl. Phys. **18**, 667–680 (1947).

Two methods are available for calculating the diffraction of the electromagnetic field of a Hertzian dipole around the earth: (i) the ray method which is valid in the lit region, (ii) the method of normal modes which is suited for the shadow region beyond the horizon of the transmitter. On the horizon and beyond it (i) breaks down, while (ii) becomes cumbersome as one recedes from the horizon into the lit region [cf. B. van der Pol and H. Bremmer, Philos. Mag. (7) **27**, 261–275 (1939)]. The author develops a theory of diffraction that is most accurate in the vicinity of the horizon while becoming less accurate outside it. Therefore, the new theory bridges the gap between (i) and (ii). First of all, the problem of diffraction around the spherical earth through the homogeneous atmosphere is transformed into that over a flat earth through an atmosphere of suitably varying index of refraction [J. C. Schelleng, C. R. Burrows and E. B. Ferrell, Proc. Inst. Radio Engrs. **21**, 427–463 (1933)]. Then conventional procedures [H. Lamb, Philos. Trans. Roy. Soc. London. Ser. A. **203**, 1–42 (1904); A. Sommerfeld in Frank and von Mises, "Die Differential- und Integralgleichungen der Mechanik und Physik," 2d ed., Braunschweig, 1935; New York, 1943, v. 2, p. 918] lead to an integral representation of the wave potential, involving cylinder functions of orders $\pm\frac{1}{3}$, $\pm\frac{2}{3}$, which reduces to Sommerfeld's solution when the radius of the sphere tends to infinity. Upon introducing the asymptotic expressions for the cylinder functions, the author approximates the integrals by the saddle-point method, so as to obtain the geometric-optical approximation which is valid for short waves inside the lit region. Other approximations introduced into the integrals lead to expressions valid in the vicinity of the horizon, for transmitter/receiver either on the ground or elevated. The final formulas involve Fresnel integrals together with auxiliary functions tabulated by the author. Numerical results are illustrated by graphs and are compared with the exact values of van der Pol and Bremmer's papers.

C. J. Bouwkamp (Eindhoven).

Epstein, Paul S. Radio-wave propagation and electromagnetic surface waves. Proc. Nat. Acad. Sci. U. S. A. **33**, 195–199 (1947).

Rediscussion of the problem of surface waves in radio-wave propagation over the plane earth [A. Sommerfeld, Ann. Physik (4) **28**, 665–736 (1909); **62**, 95–96 (1920); **81**, 1135–1153 (1926); H. Weyl, Ann. Physik (4) **60**, 481–500 (1919); **62**, 482–484 (1920)]. The author's conclusions are misleading. To see this, the following comments may suffice; the notation is that of the author.

Sommerfeld's potential functions [Frank and von Mises, Die Differential- und Integralgleichungen der Mechanik und Physik, 2d ed., Braunschweig, 1935, v. 2, pp. 918–940; equation (18) on p. 927]

$$(1)\qquad \begin{aligned} \Pi &= \int_{-\infty}^{\infty} k'^2\{k^2\sigma'+k'^2\sigma\}^{-1}H(\lambda r)\exp(-\sigma z)\lambda d\lambda, \quad z>0,\\ \Pi' &= \int_{-\infty}^{\infty} k^2\{k^2\sigma'+k'^2\sigma\}^{-1}H(\lambda r)\exp(\sigma' z)\lambda d\lambda, \quad z<0, \end{aligned}$$

satisfy (i) the corresponding wave equations, (ii) the appropriate conditions at infinity, (iii) the required boundary conditions at the surface of the earth ($k^2\Pi=k'^2\Pi'$, $\partial(\Pi-\Pi')/\partial z=0$ at $z=0$), and last but not least, (iv) they both show the prescribed singularity at the origin, namely, $\Pi-2k'^2/(k^2+k'^2)R$ and $\Pi'-2k^2/(k^2+k'^2)R$ finite for $R\to 0$. In particular, condition (iv) has to be explicitly verified. This was not emphasized by Sommerfeld in Frank and von Mises's book, but he did so in his original paper, in which he uses a slightly different notation. Epstein's functions (4) and (5) [path of integration L] are claimed to be identical with Sommerfeld's system (1). This is true if, and only if,

$$1+f(\lambda)=(k'^2/k^2)f'(\lambda),$$
$$-f(\lambda)=(k^2\sigma'-k'^2\sigma)/(k^2\sigma'+k'^2\sigma),$$

instead of Epstein's equation (6):

$$1+f(\lambda)=(k^2/k'^2)f'(\lambda)=(k^2\sigma'-k'^2\sigma)/(k^2\sigma'+k'^2\sigma).$$

Let us assume that this discrepancy is due to misprints. Then the L-solution of Epstein is identical with (1). Epstein's L'-solution differs from (1) in that the path of integration is displaced over the pole of $1+f(\lambda)$. After this displacement, the conditions (i), (ii), (iii) remain fulfilled, but (iv) is no longer satisfied. This has been overlooked by the author.

C. J. Bouwkamp (Eindhoven).

Gogoladze, V. On the propagation of radio waves in the problem of A. Sommerfeld. Acad. Sci. USSR. J. Phys. **11**, 161–162 (1947).

Translation of Bull. Acad. Sci. URSS. Sér. Géograph. Géophys. [Izvestia Akad. Nauk SSSR] **10**, 115–120 (1946); these Rev. **8**, 300.

Parodi, Maurice. Propagation sur une ligne quelconque dont les paramètres, fonctions de l'espace, satisfont en chaque point à une condition analogue à celle de non-déformation. Rev. Gén. Électricité **55**, 414–415 (1946).

De Fassi, Giovanni. Sul calcolo della forma d'onda di un generatore di impulsi di tensione. Ist. Veneto Sci. Lett. Arti. Parte II. Cl. Sci. Mat. Nat. **102**, 483–518 (1 plate) (1943).

The author develops a method of solving a multiple surge voltage generator system by employing the operational technique. After reducing the inhomogeneous equations of

the system (for the charges) into a matrix equation $F_nJ=K$,

$$F_n=\begin{bmatrix} \varphi(\Delta)-(c_1\Delta)^{-1} & -(c_1\Delta)^{-1} & 0 & \cdots & 0 \\ -(c_1\Delta)^{-1} & \varphi(\Delta)-(c_1\Delta)^{-1} & 0 & \cdots & 0 \\ & \cdots & \cdots & & \\ & \cdots & \cdots & & -(c_1\Delta)^{-1}\psi(\Delta) \end{bmatrix},$$

$$\varphi(\Delta)=L\Delta+R+\frac{c_1+2c}{c_1c\Delta},\quad \psi(\Delta)=-L_n\Delta+R_n+\frac{c_1+c_n}{c_1c_n\Delta},$$

$$J=(i_1,\cdots,i_n),\quad c_1K=(h_{01},h_{12},\cdots,h_{n-1,n}),$$

$$h_{kl}=h_k-h_l,\ h_0=h_n=0,$$

he obtains the current components. The currents i_k are calculated explicitly for a double and triple system. Next, the matrix equation for the homogeneous differential equations (for the currents i_k) is derived and the characteristic determinant F_n is expressed in terms of a derived determinant H_n, which satisfies the difference equation $H_n=\varphi(\Delta)H_{n-1}-H_{n-2}$ and $F_n=H_n-H_{n-1}$. By setting $H_n=cx^n$ and $\varphi(\Delta)=2\cosh\vartheta$, H_n is found to be a linear combination of $\sinh\vartheta$ and $\cosh\vartheta$. To evaluate ϑ from the characteristic determinant $F_n=0$ the author first assumes $\varphi(\Delta)=\psi(\Delta)$ (physically plausible). This gives imaginary values to ϑ. Then he calculates the values of $\varphi(\Delta)$ for a triple system, which agree to within 1% with the values obtained differently in the previous examples. Finally he applies the above analysis to an actual quadruple circuit, in which $\varphi\neq\psi$. The current in the 4th circuit is calculated explicitly as function of the circuit constants and the initial conditions, the results of which (numerical and graphical) are compared with the observed oscillograms as shown in the paper. *N. Chako.*

Aymerich, Giuseppe. Sul moto di un corpuscolo elettrizzato in presenza di un dipolo magnetico nel piano equatoriale del dipolo e sui moti prossimi a questo piano. Rend. Sem. Fac. Sci. Univ. Cagliari **15** (1945), 193–215 (1947).

The author considers a magnetic dipole of finite length and studies periodic motions of an electron which occur in the equatorial plane of the dipole. Small oscillations around such motions and their stability conditions are also analyzed. *I. Opatowski* (Ann Arbor, Mich.).

Lemaître, G., et Bossy, L. Sur un cas limite du problème de Størmer. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **31** (1945), 357–364 (1946).

Study of trajectories of an electron in the neighborhood of lines of force of a magnetic dipole field. *I. Opatowski* (Ann Arbor, Mich.).

Pignedoli, Antonio. Moto di un elettrone in un campo magnetico e in un campo elettrico sovrapposti, uniformi ed uniformemente rotanti intorno ad un asse. Atti Soc. Nat. Mat. Modena (6) **77**, 40–44 (1946).

Quantum Mechanics

Castoldi, L. Causalità e indeterminazione nei fondamenti della meccanica quantica. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **2**, 610–616 (1947).

Arnous, Edmond. Lois de probabilité en mécanique ondulatoire. J. Phys. Radium (8) **8**, 87–93 (1947).

The author asserts that in order to combine rigor with computational facility in determining the probability distributions of the spectral values of an observable A, it is advantageous to consider in place of A the one-parameter continuous group $U_t=\exp(itA)$, $-\infty<t<\infty$. He notes that the characteristic function of the probability distribution of A for the state with wave function ψ is simply $(U_t\psi,\psi)$ and in further support of his assertion discusses several special situations. *I. E. Segal* (Chicago, Ill.).

Forbat, N. Sur la séparation des variables dans l'équation de Schrödinger. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **32** (1946), 258–265 (1947).

For a general mechanical system in curvilinear coordinates, the author finds conditions which the kinetic and potential energy functions must satisfy in order that the Schrödinger equation for the system may be solvable by the method of separation of variables. He shows that a system of Liouville type always satisfies these conditions, and then finds a more general type of system which also allows the separation of variables. This latter generalization is shown to contain the hydrogen atom as a special case. *O. Frink* (State College, Pa.).

Bieberman, L. M. On the theory of the diffusion of resonance radiation. Akad. Nauk SSSR. Zhurnal Eksper. Teoret. Fiz. **17**, 416–426 (1947). (Russian. English summary)

In this paper the problem of the emission of resonance lines by the combined effects of excitation by radiation and quenching by collisions is considered. For the number of excited atoms $n_a(x)$ at a point x in a plane-parallel medium, the integral equation

$$(*)\qquad n_a(x)=\lambda\int_0^1 n_a(\xi)K(|\xi-x|)d\xi+\lambda\eta B(x)$$

is derived, where

$$K(y)=\tfrac{1}{2}k_0l\pi^{-\frac{1}{2}}\int_1^\infty\int_{-\infty}^\infty u^{-1}\exp\{-2\omega^2-k_0lye^{-\omega^2}\}du\,d\omega,$$

λ is related to the mean life τ in the excited state and the coefficient of collisional quenching, k_0l is the optical thickness of the medium at the center of the line and $B(x)$ is a certain known function. The integral equation (*) is solved by a numerical process which involves its replacement by a system of linear equations. *S. Chandrasekhar.*

Broch, Einar Klaumann. Two-body inverse-square central field problems in relativistic quantum mechanics. A contribution to the theory of the hydrogen atom. Arch. Math. Naturvid. **48**, no. 1, 1–23 (1945).

The author solves the radial Dirac equations for an electron in an inverse-square central force field which changes from an attraction for $r>r_0$ to a repulsion for $r<r_0$, where r_0 is the radius of the nucleus. The solutions involve confluent hypergeometric functions and the related Whittaker functions. Different formulas are required for the two regions $r>r_0$ and $r<r_0$, and these must be matched at the boundary $r=r_0$. From these results the author derives a correction to the Sommerfeld fine-structure formula. He concludes that if the repulsive force is assumed to be large enough, this correction can account for observed deviations of certain spectral lines of deuterium from values predicted by the Sommerfeld formula, although Sommerfeld considered these deviations to be inconsistent with the Dirac theory. *O. Frink* (State College, Pa.).

Broch, Einar Klaumann. On the evaluation of the isotope shift in hyperfine structure. Arch. Math. Naturvid. **48**, no. 2, 25–35 (1945).

For an atomic nucleus of radius r_0, a potential $V(r)$ is assumed which is of Coulomb type for $r>r_0$, but not for $r<r_0$. Using perturbation methods applied to the Dirac equations of an electron in such a field, the author finds the change in energy due to a change in the nuclear radius r_0. Observed differences in the hyperfine structure of the spectra of different isotopes of the same element may be explainable as due to such a change in radius. The formulas obtained involve Bessel functions of nonintegral order. The author's numerical results agree, for atoms of small atomic number, with those of Breit and Rosenthal, which were derived on a slightly different assumption. For large atomic numbers a correction factor of the order of $\frac{1}{2}$ must be applied to the formula of Breit and Rosenthal to give the author's value.
O. Frink (State College, Pa.).

Rosen, Nathan. Statistical geometry and fundamental particles. Physical Rev. (2) **72**, 298–303 (1947).

A lower limit to the inaccuracy in measuring a space coordinate of e.g. an electron is given by assuming the results of measurement to have a Gaussian distribution with standard deviation a; time is not treated in this way. With an abstract space of points is associated an "observable" space of volumes of order of magnitude a^3 for each point. Only mean values of physical quantities over such an elementary volume are given a meaning. Physical laws, e.g. linear equations, are supposed valid in the abstract space, relations between observable quantities being obtainable by averaging. Lorentz transformations for space and time coordinates remain valid in observable space. An electron is represented by a point in abstract space. The Coulomb interaction energy of two particles remains finite as their separation vanishes: static self-energy is finite. The perturbation calculation of dynamic self-energy is divergent although each term is finite. For nucleons interacting through a meson field the first approximation to the static interaction does not have a singularity. *C. Strachan.*

Wessel, Walter. Zur Theorie des Elektrons. Z. Naturforschung 1, 622–636 (1946).

For a moving charge extended in space, radiation reaction may be expressed as an ascending series of time-derivatives of the coordinates. These higher derivatives are replaced by new variables grouped together and interpreted as tensor components: the equations of motion for these variables must be such that, by elimination of the new variables, the equation of motion of a particle subject to radiation reaction is obtained. The classical electron is described by four spinor components ψ_1, ψ_2, χ_1, χ_2 and their complex conjugates, treated not as functions of coordinates but as additional parameters of the motion. Two velocities $\mathbf{u}$ and $\mathbf{U}$ are derived from the two four-vectors $j_{\dot{m}l}=\psi_{\dot{m}}\psi_l+\chi_{\dot{m}}\chi_l$ and $k_{\dot{m}l}=\psi_{\dot{m}}\psi_l-\chi_{\dot{m}}\chi_l$ [cf. Laporte and Uhlenbeck, Physical Rev. (2) **37**, 1380–1397 (1931)]: the additional variables have spinor definitions. The Hamiltonian is cp^4, where, in the absence of external fields, $\mathbf{p}\cdot\mathbf{u}+p_4u^4=-mc$ an invariant, p^l being the momentum-energy vector. A similar invariant involving $\mathbf{U}$ gives rise to another mass term M. The attempt to eliminate all variables apart from space coordinates from the equations and so to obtain the usual equation with radiation reaction leads to difficulties even in the absence of external fields: suggestions are made to overcome these by the use of the second "mass" M to account for the emitted energy but proof of their effectiveness is lacking.
C. Strachan (Aberdeen).

Ashauer, S. On the self-accelerating electron. Proc. Cambridge Philos. Soc. **43**, 506–510 (1947).

As the equations of motion of an electron in an electromagnetic field, according to Dirac's classical theory [Proc. Roy. Soc. London. Ser. A. **167**, 148–169 (1938)] are of the third order, extra conditions have been imposed to separate the physical from the nonphysical solutions. The author examines the nonphysical solutions in the case of no external field when the velocity of the electron is close to that of light, and plots the surfaces of constant retarded scalar potential. It is found that the energy density of the field due to the electron is appreciable only within a thin disc-like region perpendicular to the orbit and just in front of the particle. *H. C. Corben* (Pittsburgh, Pa.).

de Wet, J. S. Symmetric energy-momentum tensors in relativistic field theories. Proc. Cambridge Philos. Soc. **43**, 511–520 (1947).

An exhaustive treatment of the problem of defining a symmetrical energy-momentum tensor for field theories in which (a) the Lagrangian involves only first derivatives of the field variables and (b) second and higher order derivatives appear in L. In each case, the field variables considered may be, in addition to a metric tensor $g_{\mu\nu}$, a set of scalar, vector, tensor and spinor fields. It is shown that, in the absence of spinor fields, but otherwise generally, the symmetric energy tensor is

$$T^{\mu\nu}=\frac{\partial L}{\partial g_{\mu\nu}}-\frac{\partial}{\partial x^\sigma}\left(\frac{\partial L}{\partial g_{\mu\nu,\sigma}}\right)+(\mu,\nu\text{ interchanged})$$

and that in the presence of spinor fields terms of the form obtained by Belinfante [Physica **6**, 887–898 (1939); these Rev. **1**, 280] must be added to this. *H. C. Corben.*

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Volume 1, number 1 appeared in 1947. The journal is published in Wrocław.

Mathematics Magazine.

This continues the National Mathematics Magazine. It is published in Los Angeles. Volume **21**, number 1, is dated September–October, 1947.